

# Fourier transform for $D$ -algebras

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This paper is devoted to the construction of an analogue of the Fourier transform for a certain class of non-commutative algebras. The model example which initiated this study is the equivalence between derived categories of  $\mathcal{D}$ -modules on an abelian variety and  $\mathcal{O}$ -modules on the universal extension of the dual abelian variety by a vector space (see [L], [R2]). The natural framework for a generalization of this equivalence is provided by the language of  $D$ -algebras developed by A. Beilinson and J. Bernstein in [BB]. We consider a subclass of  $D$ -algebras we call special  $D$ -algebras. We show that whenever one has an equivalence of categories of  $\mathcal{O}$ -modules on two varieties  $X$  and  $Y$ , it gives rise to a correspondence between special  $D$ -algebras on  $X$  and  $Y$  such that the corresponding derived categories of modules are equivalent. When  $X$  is an abelian variety,  $Y$  is the dual abelian variety, according to Mukai [M] the categories of  $\mathcal{O}$ -modules on  $X$  and  $Y$  are equivalent, so our construction gives in particular the Fourier transform between modules over rings of twisted differential operators (*tdo*, for short, see section 1 for a definition) with non-degenerate first Chern class on  $X$  and  $Y$ .

We also deal with the microlocal version of the Fourier transform. The microlocalization of a special filtered  $D$ -algebra on  $X$  is an NC-scheme in the sense of Kapranov (see [K]) over  $X$ , i.e. a ringed space whose structure ring is complete with respect to the topology defined by commutator filtration (see section 3). We show that in our situation the derived categories of coherent sheaves on microlocalizations are also equivalent. In the case of rings of twisted differential operators on the dual abelian varieties  $X$  and  $Y$  one can think about the corresponding microlocalized algebras as deformation quantizations of the cotangent spaces  $T^*X$  and  $T^*Y$ . The projections  $p_X : T^*X \rightarrow T_0^*X$  and  $p_Y : T^*Y \rightarrow T_0^*Y$  can be considered as completely integrable systems with dual fibers (a choice of a non-degenerate *tdo* on  $X$  induces an identification of the bases  $T_0^*X$  and  $T_0^*Y$  of these systems). We conjecture that our construction generalizes to other dual completely integrable systems. This hope is based on the following observation: the relative version of our transform gives a Fourier transform for modules over relative *tdo*'s on dual families of abelian varieties, while deformation quantizations are usually subalgebras in these *tdo*'s. An example of dual completely integrable systems appears in the geometric Langlands program (see [BD]). Namely, one may consider Hitchin systems for Langlands dual groups. The analogue of the Fourier transform in this situation should lead to the equivalence between modules over a microlocalized *tdo* on moduli spaces of principal bundles for Langlands dual groups.

An important aspect of our work is that in an appropriate sense, the microlocalized Fourier transform is in fact étale local. Given a special filtered  $D$ -algebra

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$\mathcal{A}$  on  $X$ , let  $\mathcal{A}_{ml}$  be the corresponding microlocalized scheme. Then  $\mathcal{A}_{ml}$  is a non-commutative thickening of the product of  $X$  with a scheme  $Z$ . Denoting by  $\Phi(\mathcal{A})$  the corresponding  $D$ -algebra on  $Y$ ,  $\Phi(\mathcal{A})_{ml}$  is a thickening of  $Y \times Z$ . We then prove that the microlocalized Fourier transform is étale local in  $Z$ . It should be noted that  $\mathcal{A}_{ml}$  and  $\Phi(\mathcal{A})_{ml}$  are not schemes over  $Z$ , so one does not have a straightforward base-change argument. Rather, we develop in section 3 the non-commutative version of the theory of étale morphisms in the framework of Kapranov's NC-schemes, and establish the equivalence by a version of the topological invariance of étale morphisms.

Our work is motivated in part by Krichever's construction of solutions of the KP-hierarchy, [Kr]. (See section 5.) Let  $W$  be a smooth variety of dimension  $r$ , embedded in its Albanese variety,  $X$ . Let  $D \subset W$  be an ample hypersurface and  $V \subset H^0(D, \mathcal{O}_D(D))$  an  $r$ -dimensional basepoint free subspace. Let  $\phi : D \rightarrow \mathbb{P}(V^*)$  denote the corresponding morphism and let  $U \subset D$  be an open subset such that  $\phi|_U$  is étale. Let  $Y = \text{Pic}^0(W)$ . Then  $V$  maps to the space of vector fields on  $Y$ , and hence one has a subalgebra of the differential operators on  $Y$ , consisting of those operators which differentiate only in the “ $V$ ” directions. Denote this algebra by  $\Phi(\mathcal{A}_V)$ . That is,  $\Phi(\mathcal{A}_V)$  is dual to a  $D$ -algebra  $\mathcal{A}_V$  on  $X$ . Then the microlocalizations of these  $D$ -algebras are thickenings of  $Y \times \mathbb{P}(V^*)$  and  $X \times \mathbb{P}(V^*)$  respectively. In particular, one has the étale localizations  $\mathcal{A}_{ml,U}$  and  $\Phi(\mathcal{A})_{ml,U}$  supported on  $X \times U$  and  $Y \times U$  respectively. Let  $U_\infty$  denote the formal neighborhood of  $U$  in  $W$ . Then the diagonal embedding  $U \rightarrow X \times U$  extends to an embedding  $\Delta_\infty : U_\infty \rightarrow \mathcal{A}_{ml,U}$ . On the other hand,  $U \times Y$  sits in both  $X \times Y$  and  $\Phi(\mathcal{A})_{ml,U}$ . Denote by  $\mathcal{L}_\infty$  the Fourier transform of  $\Delta_{\infty*}(\mathcal{O}_{U_\infty})$ . We prove that  $\mathcal{L}_\infty$  is a locally free rank-one left  $\mathcal{O}$  module on  $\Phi(\mathcal{A})_{ml,U}$ , whose restriction to  $U \times Y$  is the restriction of the Poincaré line bundle. Thus  $\mathcal{L}_\infty$  is a deformation of the Poincaré line bundle. Furthermore, for any positive integer  $k$ , the Fourier transform of  $\Delta_{\infty*}(\mathcal{O}_{U_\infty}(kU))$  is  $\mathcal{L}_\infty(k(U \times Y))$ .

The point is that the ring  $H^0(W, \mathcal{O}(*D))$  acts by  $\mathcal{A}_{ml,U}$ -endomorphisms on  $\Delta_{\infty*}(\mathcal{O}_{U_\infty}(*U))$ . Functoriality of the Fourier transform then gives us a representation

$$(0.0.1) \quad H^0(W, \mathcal{O}(*D)) \rightarrow \text{End}_{\Phi(\mathcal{A})_{ml,U}}(\mathcal{L}_\infty(*U \times Y)) .$$

When  $W$  is a curve and  $D$  is a point, this representation reduces to the Burchall-Chaundy [BC] representation of  $H^0(W, \mathcal{O}(*D))$  by differential operators. We intend to study the representation (0.0.1) further in a future work. In particular, the problem of characterizing the image of this representation is quite interesting, and should lead to generalizations of the KP-hierarchy.

*Notation.* Fix a scheme  $S$ . Given an  $S$ -scheme  $U$ , denote by  $\pi_S^U$  the structural morphism. By “associative  $S$ -algebra on  $U$ ” we mean a sheaf of associative rings  $\mathcal{A}$  on  $U$  equipped with a morphism of sheaves of rings from  $\pi_S^{U^{-1}}(\mathcal{O}_S)$  to the center of  $\mathcal{A}$ . We abbreviate “ $\otimes_{\pi_S^{U^{-1}}(\mathcal{O}_S)}$ ” by “ $\otimes_S$ ”. For a scheme  $U$  we denote by  $\mathcal{D}^b(U)$  the bounded derived category of quasicoherent sheaves on  $U$ . Throughout the paper,  $X$  and  $Y$  are flat, separated  $S$ -schemes.

1.  $D$ -ALGEBRAS AND LIE ALGEBROIDS

**1.1.** Let us recall some definitions from [BB]. A *differential  $\mathcal{O}_X$ -bimodule*  $M$  is a quasicoherent sheaf on  $X \times_S X$  supported on the diagonal  $X \subset X \times_S X$ . One can consider the category of differential  $\mathcal{O}_X$ -bimodules as a subcategory in the category of all sheaves of  $\mathcal{O}_X$ -bimodules on  $X$ . A  *$D$ -algebra* on  $X$  is a sheaf of flat, associative  $S$ -algebras  $\mathcal{A}$  on  $X$  equipped with a morphism of  $S$ -algebras  $i : \mathcal{O}_X \rightarrow \mathcal{A}$  such that  $\mathcal{A}$  is a differential  $\mathcal{O}_X$ -bimodule. This means that  $\mathcal{A}$  has an increasing filtration  $0 = \mathcal{A}_{-1} \subset \mathcal{A}_0 \subset \mathcal{A}_1 \subset \dots$  such that  $\mathcal{A} = \cup \mathcal{A}_n$  and  $ad(f)(\mathcal{A}_k) \subset \mathcal{A}_{k-1}$  for any  $k \geq 0$  and  $f \in \mathcal{O}_X$  where  $ad(m) := rm - mr$ . We denote by  $b(\mathcal{A})$  the quasi-coherent sheaf on  $X \times_S X$  (supported on the diagonal) corresponding to  $\mathcal{A}$ . Also we denote by  $\mathcal{M}(\mathcal{A})$  the category of sheaves of left  $\mathcal{A}$ -modules on  $X$  which are quasicoherent as  $\mathcal{O}_X$ -modules.

**1.2.** Let us describe some basic operations with  $D$ -algebras and modules over them. Let  $\mathcal{A}_X$  and  $\mathcal{A}_Y$  be  $D$ -algebras over  $X$  and  $Y$  respectively. One defines a  $D$ -algebra  $\mathcal{A}_X \boxtimes_S \mathcal{A}_Y$  on  $X \times_S Y$  by gluing  $D$ -algebras over products of affine opens  $U \times_S V$  corresponding to  $\mathcal{A}_X(U) \otimes_S \mathcal{A}_Y(V)$ . A module  $M \in \mathcal{M}(\mathcal{A}_X \boxtimes_S \mathcal{A}_Y)$  is the same as a quasicoherent  $\mathcal{O}_{X \times_S Y}$ -module together with commuting actions of  $p_X^{-1}(\mathcal{A}_X)$  and  $p_Y^{-1}(\mathcal{A}_Y)$  which are compatible with the  $\mathcal{O}_{X \times_S Y}$ -module structure (where  $p_X$  and  $p_Y$  are projections from  $X \times_S Y$  to  $X$  and  $Y$ ). In particular, we have the natural structure of  $D$ -algebra on  $p_X^* \mathcal{A}_X \simeq \mathcal{A}_X \boxtimes_S \mathcal{O}_Y$  and  $p_Y^* \mathcal{A}_Y \simeq \mathcal{O}_X \boxtimes_S \mathcal{A}_Y$  and natural embeddings of  $D$ -algebras  $p_X^* \mathcal{A}_X \hookrightarrow \mathcal{A}_X \boxtimes_S \mathcal{A}_Y$ ,  $p_Y^* \mathcal{A}_Y \hookrightarrow \mathcal{A}_X \boxtimes_S \mathcal{A}_Y$ . For a pair of modules  $M_X \in \mathcal{M}(\mathcal{A}_X)$  and  $M_Y \in \mathcal{M}(\mathcal{A}_Y)$  there is a natural structure of  $\mathcal{A}_X \boxtimes_S \mathcal{A}_Y$ -module on  $M_X \boxtimes_S M_Y$ .

Now assume that we have  $D$ -algebras  $\mathcal{A}_X$ ,  $\mathcal{A}_Y$ , and  $\mathcal{A}_Z$  on  $X$ ,  $Y$  and  $Z$  respectively. Then we can define an operation

$$\circ_{\mathcal{A}_Y} : \mathcal{D}^-(\mathcal{M}(\mathcal{A}_X \boxtimes_S \mathcal{A}_Y^{op})) \times \mathcal{D}^-(\mathcal{M}(\mathcal{A}_Y \boxtimes_S \mathcal{A}_Z)) \rightarrow \mathcal{D}^-(\mathcal{M}(\mathcal{A}_X \boxtimes_S \mathcal{A}_Z)).$$

The definition is the globalization of the operation of tensor product of bimodules. Namely, for a pair of objects  $M \in \mathcal{D}^-(\mathcal{M}(\mathcal{A}_X \boxtimes_S \mathcal{A}_Y^{op}))$  and  $N \in \mathcal{D}^-(\mathcal{M}(\mathcal{A}_Y \boxtimes_S \mathcal{A}_Z))$  we can form the external tensor product  $M \boxtimes_S N \in \mathcal{D}^-(\mathcal{M}(\mathcal{A}_{XYZ}))$  where  $\mathcal{A}_{XYZ} = \mathcal{A}_X \boxtimes_S \mathcal{A}_Y^{op} \boxtimes_S \mathcal{A}_Y \boxtimes_S \mathcal{A}_Z$  is a  $D$ -algebra on  $X \times_S Y \times_S Y \times_S Z$ . Note that there is a natural structure of left  $\mathcal{A}_Y \boxtimes_S \mathcal{A}_Y^{op}$ -module on  $b(\mathcal{A}_Y)$  given by the multiplication in  $\mathcal{A}_Y$ . Hence, we can consider the tensor product

$$(\mathcal{A}_X \boxtimes_S b(\mathcal{A}_Y) \boxtimes_S \mathcal{A}_Z) \stackrel{\mathbb{L}}{\otimes}_{\mathcal{A}_{XYZ}} (M \stackrel{\mathbb{L}}{\boxtimes}_S N)$$

as an object in the category  $\mathcal{D}^-(p_{XZ}^{-1}(\mathcal{A}_X \boxtimes_S \mathcal{A}_Z))$  where  $p_{XZ}^{-1}$  denotes a sheaf-theoretical inverse image. Finally, we set

$$M \circ_{\mathcal{A}_Y} N = Rp_{XZ*}((\mathcal{A}_X \boxtimes_S b(\mathcal{A}_Y) \boxtimes_S \mathcal{A}_Z) \stackrel{\mathbb{L}}{\otimes}_{\mathcal{A}_{XYZ}} (M \stackrel{\mathbb{L}}{\boxtimes}_S N)).$$

There is also the following equivalent definition:

$$M \circ_{\mathcal{A}_Y} N = Rp_{XZ*}((M \boxtimes_S \mathcal{A}_Z) \stackrel{\mathbb{L}}{\otimes}_{\mathcal{A}_X^{op} \boxtimes_S \mathcal{A}_Y \boxtimes_S \mathcal{A}_Z} (\mathcal{A}_X^{op} \boxtimes_S N)).$$

Specializing to the case that  $Z = S$  and  $\mathcal{A}_Z = \mathcal{O}_S$ , we see that every  $\mathcal{A}_X \boxtimes_S \mathcal{A}_Y^{op}$ -module  $F$  on  $X \times_S Y$  defines a functor  $G \mapsto F \circ_{\mathcal{A}_Y} G$  from  $\mathcal{D}^-\mathcal{M}(\mathcal{A}_Y)$  to  $\mathcal{D}^-\mathcal{M}(\mathcal{A}_X)$ .

**Proposition 1.1.** (1) *The operation  $\circ$  is associative in the natural sense.*

- (2) *One has  $M \circ b(\mathcal{A}_Y) \simeq M$  and  $b(\mathcal{A}_Y) \circ N \simeq N$  canonically, for  $M \in \mathcal{D}^-\mathcal{M}(\mathcal{A}_X \boxtimes_S \mathcal{A}_Y^{op})$  and  $N \in \mathcal{D}^-\mathcal{M}(\mathcal{A}_Y \boxtimes_S \mathcal{A}_Z)$ .*
- (3) *If  $M$  is a differential  $\mathcal{O}_X$ -bimodule and  $N$  is an  $\mathcal{O}_{X \times_S Y}$ -module, then  $b(M) \circ_{\mathcal{O}_X} N = p_X^{-1}(M) \otimes_{p_X^{-1}(\mathcal{O}_X)} N$ .*

It follows that if  $\mathcal{A}$  is a  $D$ -algebra on  $X$ , the structural morphism on  $\mathcal{A}$  may be viewed as a morphism  $b(\mathcal{A}) \circ_{\mathcal{O}_X} b(\mathcal{A}) \rightarrow b(\mathcal{A})$ , and if  $M$  is a left  $\mathcal{A}$ -module, the action of  $\mathcal{A}$  on  $M$  is given by a morphism  $b(\mathcal{A}) \circ_{\mathcal{O}_X} M \rightarrow M$ . Moreover, an  $\mathcal{A}_X \boxtimes_S \mathcal{A}_Y^{op}$ -module structure on an  $\mathcal{O}_{X \times_S Y}$ -module  $M$  is the same as a pair of morphisms  $b(\mathcal{A}_X) \circ_{\mathcal{O}_X} M \rightarrow M$  and  $M \circ_{\mathcal{O}_Y} b(\mathcal{A}_Y) \rightarrow M$  making  $M$  a (left  $b(\mathcal{A}_X)$ )-(right  $b(\mathcal{A}_Y)$ )-module with respect to  $\circ$ , such that the two module structures commute.

**1.3.** Recall that a *Lie algebroid*  $L$  on  $X$  is a (quasicoherent)  $\mathcal{O}_X$ -module equipped with a morphism of  $\mathcal{O}_X$ -modules  $\sigma : L \rightarrow \mathcal{T} := \text{Der}_S \mathcal{O}_X$  (=relative tangent sheaf of  $X$ ) and an  $S$ -linear Lie bracket  $[\cdot, \cdot] : L \otimes_S L \rightarrow L$  such that  $\sigma$  is a homomorphism of Lie algebras and the following identity is satisfied:

$$[\ell_1, f\ell_2] = f \cdot [\ell_1, \ell_2] + \sigma(\ell_1)(f)\ell_2$$

where  $\ell_1, \ell_2 \in L, f \in \mathcal{O}_X$ . To every Lie algebroid  $L$  one can associate a  $D$ -algebra  $\mathcal{U}(L)$  called the *universal enveloping algebra* of  $L$ . By definition  $\mathcal{U}(L)$  is a sheaf of algebras equipped with the morphisms of sheaves  $i : \mathcal{O}_X \rightarrow \mathcal{U}(L), i_L : L \rightarrow \mathcal{U}(L)$ , such that  $\mathcal{U}(L)$  is generated, as an algebra, by the images of these morphisms and the only relations are:

- (i)  $i$  is a morphism of algebras;
- (ii)  $i_L$  is a morphism of Lie algebras;
- (iii)  $i_L(f\ell) = i(f)i_L(\ell), [i_L(\ell), i(f)] = i(\sigma(\ell)(f)),$  where  $f \in \mathcal{O}_X, \ell \in L$ .

**1.4.** Let  $L$  be a Lie algebroid on  $X$ . A central extension of  $L$  by  $\mathcal{O}_X$  is a Lie algebroid  $\tilde{L}$  on  $X$  equipped with an embedding of  $\mathcal{O}_X$ -modules  $c : \mathcal{O}_X \hookrightarrow \tilde{L}$  such that  $[c(1), \tilde{\ell}] = 0$  for every  $\tilde{\ell} \in \tilde{L}$  (in particular,  $c(\mathcal{O}_X)$  is an ideal in  $\tilde{L}$ ), and an isomorphism of Lie algebroids  $\tilde{L}/c(\mathcal{O}_X) \simeq L$ . For such a central extension we denote by  $\mathcal{U}^\circ(\tilde{L})$  the quotient of  $\mathcal{U}(\tilde{L})$  modulo the ideal generated by the central element  $i(1) - i_{\tilde{L}}(c(1))$ .

**Lemma 1.1.** *Let  $L$  be a locally free  $\mathcal{O}_X$ -module of finite rank. Then there is a bijective correspondence between isomorphism classes of the following data:*

- (i) *a structure of a Lie algebroid on  $L$  and a central extension  $\tilde{L}$  of  $L$  by  $\mathcal{O}_X$ .*
- (ii) *a  $D$ -algebra  $\mathcal{A}$  equipped with an increasing algebra filtration  $\mathcal{O}_X = \mathcal{A}_0 \subset \mathcal{A}_1 \subset \mathcal{A}_2 \subset \dots$  such that  $\cup \mathcal{A}_n = \mathcal{A}$  and an isomorphism of the associated graded algebra  $\text{gr } \mathcal{A}$  with the symmetric algebra  $S^\bullet L$ .*

The correspondence between (i) and (ii) maps a central extension  $\tilde{L}$  to  $\mathcal{U}^\circ(\tilde{L})$ .

**1.5.** Assume that  $X$  is smooth over  $S$ . Then one can take  $L = \mathcal{T}$  with its natural Lie algebroid structure. The corresponding central extensions  $\tilde{\mathcal{T}}$  of  $\mathcal{T}$  by  $\mathcal{O}$  are called *Picard algebroids* and the associated  $D$ -algebras are called *algebras of twisted differential operators*; or simply *tdo's*. If  $\mathcal{D}$  is a tdo,  $\mathcal{D}_{-1} = 0 = \mathcal{D}_0 \subset \mathcal{D}_1 \subset \mathcal{D}_2 \subset \dots$  is its maximal  $D$ -filtration, i.e.

$$\mathcal{D}_i = \{d \in \mathcal{D} \mid \text{ad}(f)d \in \mathcal{D}_{i-1}, f \in \mathcal{O}_X\},$$

then  $\text{gr } \mathcal{D} \simeq S^\bullet \mathcal{T}$ .

**Lemma 1.2.** *For a locally free  $\mathcal{O}_X$ -module of finite rank  $E$  one has a canonical isomorphism*

$$\text{Ext}_{\mathcal{O}_{X \times_S X}}^1(\Delta_* E, \Delta_* \mathcal{O}_X) \simeq \text{Hom}_{\mathcal{O}_X}(E, \mathcal{T}) \oplus \text{Ext}_{\mathcal{O}_X}^1(E, \mathcal{O}_X),$$

where  $X \xrightarrow{\Delta} X \times_S X$  is the diagonal embedding.

*Proof.* Since  $\Delta_* E \simeq p_1^* E \otimes_{\mathcal{O}_{X \times_S X}} (\mathcal{O}_{X \times_S X} / J)$ , where  $J$  is the ideal sheaf of the diagonal, we have an exact sequence

$$0 \rightarrow \text{Hom}(p_1^* E \otimes_{\mathcal{O}_{X \times_S X}} J, \Delta_* \mathcal{O}_X) \rightarrow \text{Ext}^1(\Delta_* E, \Delta_* \mathcal{O}_X) \rightarrow \text{Ext}^1(p_1^* E, \Delta_* \mathcal{O}_X)$$

Note that the first and last terms are isomorphic to  $\text{Hom}(E, \mathcal{T})$  and  $\text{Ext}^1(E, \mathcal{O}_X)$  respectively. It remains to note that there is a canonical splitting  $\Delta_* : \text{Ext}^1(E, \mathcal{O}_X) \rightarrow \text{Ext}^1(\Delta_* E, \Delta_* \mathcal{O})$ .  $\square$

Note that the projection  $\text{Ext}^1(\Delta_* E, \Delta_* \mathcal{O}_X) \rightarrow \text{Hom}(E, \mathcal{T})$  can be described as follows. Given an extension

$$0 \longrightarrow \Delta_* \mathcal{O}_X \longrightarrow \tilde{E} \longrightarrow \Delta_* E \longrightarrow 0$$

the action of  $J/J^2$  on  $\tilde{E}$  induces the morphism  $J/J^2 \otimes \tilde{E} \rightarrow \Delta_* \mathcal{O}$ , which factors through  $J/J^2 \otimes \Delta_* E$ , since  $J$  annihilates  $\Delta_* \mathcal{O}$ . Hence we get a morphism  $\Delta_* E \rightarrow \Delta_* \mathcal{T}$ .

Now if  $\mathcal{A}$  is a  $D$ -algebra, equipped with a filtration  $\mathcal{A}_\bullet$  such that  $\text{gr } \mathcal{A} \simeq S^\bullet(E)$ , then we consider the corresponding extension of  $\mathcal{O}_X$ -bimodules

$$0 \longrightarrow \mathcal{O}_X = \mathcal{A}_0 \longrightarrow \mathcal{A}_1 \longrightarrow E = \mathcal{A}_1 / \mathcal{A}_0 \longrightarrow 0$$

as an element in  $\text{Ext}_{\mathcal{O}_{X \times X}}^1(\Delta_* E, \Delta_* \mathcal{O}_X)$ . By definition,  $\mathcal{A}$  is a tdo if the projection of this element to  $\text{Hom}_{\mathcal{O}_X}(E, \mathcal{T})$  is a map  $E \rightarrow \mathcal{T}$  which is an isomorphism.

## 2. EQUIVALENCES OF CATEGORIES OF MODULES OVER $D$ -ALGEBRAS

**2.1.** Let  $P$  be an object in  $\mathcal{D}^b(X \times_S Y)$ ,  $Q$  be an object in  $\mathcal{D}^b(Y \times_S X)$  such that

$$P \circ_{\mathcal{O}_Y} Q \simeq \Delta_* \mathcal{O}_Y, Q \circ_{\mathcal{O}_X} P \simeq \Delta_* \mathcal{O}_X.$$

where  $\Delta$  denotes the diagonal embedding. In this case the functors

$$\Phi_P : M \mapsto P \circ M, \Phi_Q : N \mapsto Q \circ N$$

establish an equivalence of categories  $\mathcal{D}^-(X)$  and  $\mathcal{D}^-(Y)$ .

For example, we have these data in the following situation:  $X$  is an abelian  $S$ -scheme,  $Y = \hat{X}$  is the dual abelian  $S$ -scheme,  $P = \mathcal{P}$  is the normalized Poincaré

line bundle on  $X \times_S \hat{X}$ ,  $Q = \sigma^* \mathcal{P}^{-1} \omega_X^{-1}[-g]$  where  $\sigma : \hat{X} \times_S X \rightarrow X \times_S \hat{X}$  is the permutation of factors,  $g = \dim X$ .

**2.2.** Let us call a quasicoherent sheaf  $K$  on  $X \times_S X$  *special* if there is a filtration  $0 = K_{-1} \subset K_0 \subset K_1 \subset \dots$  of  $K$  and a sequence of sheaves of flat, quasicoherent  $\mathcal{O}_S$ -modules  $F_i$  such that  $\cup K_i = K$  and  $K_i/K_{i-1} \simeq \Delta_* \pi_S^{X*}(F_i)$  for every  $i \geq 0$ . We denote by  $\mathcal{S}_X$  the exact category of special sheaves on  $X \times_S X$ . The following properties are easily verified.

**Lemma 2.1.** *Let  $F \in \mathcal{D}^-(\mathcal{M}(\mathcal{O}_S))$ .*

- (1) *Let  $G \in \mathcal{D}^-(\mathcal{M}(\mathcal{O}_{X \times_S Y}))$ . Then  $\Delta_{X*} \pi_S^{X*}(F) \circ_{\mathcal{O}_X} G = \pi_S^{X \times_S Y*}(F) \otimes_{\mathcal{O}_{X \times_S Y}}^{\mathbb{L}} G$ .*
- (2)  *$P \circ_{\mathcal{O}_Y} \pi_S^{Y*}(F) \circ_{\mathcal{O}_Y} Q = \pi_S^{X*}(F)$ .*

The following proposition then follows.

**Proposition 2.1.** (1) *For every  $K \in \mathcal{S}_Y$ , the functor  $\mathcal{M}(\mathcal{O}_{Y \times_Z}) \rightarrow \mathcal{M}(\mathcal{O}_{Y \times_Z})$ ,  $M \mapsto K \circ_{\mathcal{O}_Y} M$ , is exact.*

- (2) *For every pair of special sheaves  $K, K' \in \mathcal{S}_Y$ ,  $K \circ_{\mathcal{O}_Y} K'$  is special.*

**Proposition 2.2.** *The functor  $\Phi : K \mapsto P \circ_{\mathcal{O}_Y} K \circ_{\mathcal{O}_Y} Q$  defines an equivalence of categories  $\Phi : \mathcal{S}_Y \rightarrow \mathcal{S}_X$ .*

*Proof.* From lemma 2.1, together with the fact that the operation  $\circ$  commutes with inductive limits, we have  $\Phi(K) \in \mathcal{S}_X$  for every  $K \in \mathcal{S}_Y$ . It remains to notice that there is an inverse functor to  $\Phi$  given by

$$\Phi^{-1}(K') = Q \circ_{\mathcal{O}_X} K' \circ_{\mathcal{O}_X} P$$

where  $K' \in \mathcal{S}_X$ .  $\square$

**Proposition 2.3.** *For  $K, K' \in \mathcal{S}_Y, M \in \mathcal{D}^b(Y)$  one has a canonical isomorphism of  $\mathcal{O}_X$ -bimodules*

$$\Phi(K \circ_{\mathcal{O}_Y} K') \simeq \Phi K \circ_{\mathcal{O}_X} \Phi K',$$

*and a canonical isomorphism in  $\mathcal{D}^b(X)$*

$$\Phi(K \circ_{\mathcal{O}_Y} M) \simeq \Phi K \circ_{\mathcal{O}_X} \Phi M.$$

**Definition 2.1.** A *special  $D$ -algebra* on  $X$  is a  $D$ -algebra  $\mathcal{A}$  such that the sheaf  $b(\mathcal{A})$  on  $X \times_S X$  is special.

It follows from the above proposition that for any special  $D$ -algebra  $\mathcal{A}$  on  $Y$  there exists a canonical  $D$ -algebra  $\Phi \mathcal{A}$  on  $X$  such that

$$b(\Phi \mathcal{A}) \simeq \Phi(b(\mathcal{A})).$$

Namely one just has to apply  $\Phi$  to structural morphisms  $b(\mathcal{A}) \circ_{\mathcal{O}_Y} b(\mathcal{A}) \rightarrow b(\mathcal{A})$  and  $\Delta_* \mathcal{O}_Y \rightarrow b(\mathcal{A})$ . Furthermore, we now prove that the derived categories of modules over  $\mathcal{A}$  and  $\Phi \mathcal{A}$  are equivalent.

**Theorem 2.1.** *Assume that  $P$  and  $Q$  are quasi-coherent sheaves up to a shift (i.e. they have only one cohomology). Then for every special  $D$ -algebra  $\mathcal{A}$  on  $Y$  there is a canonical exact equivalence  $\Phi : \mathcal{D}^- \mathcal{M}(\mathcal{A}) \rightarrow \mathcal{D}^- \mathcal{M}(\Phi \mathcal{A})$  such that the following diagram of functors is commutative:*

$$\begin{array}{ccc}
\mathcal{D}^b \mathcal{M}(\mathcal{A}) & \xrightarrow{\Phi} & \mathcal{D}^b \mathcal{M}(\Phi \mathcal{A}) \\
\downarrow & & \downarrow \\
\mathcal{D}^b(Y) & \xrightarrow{\Phi} & \mathcal{D}^b(X)
\end{array}$$

where the vertical arrows are the forgetting functors.

*Proof.* Let us consider the following object in  $\mathcal{D}^b(X \times_S Y)$ :

$$\mathcal{B} = P \circ_{\mathcal{O}_Y} b(\mathcal{A}) = P \otimes_{\mathcal{O}_{X \times_S Y}} p_Y^* \mathcal{A}.$$

Note that  $\mathcal{B}$  is actually concentrated in one degree so we can consider it as a quasicoherent sheaf on  $X \times_S Y$  (perhaps shifted). We claim that there is a canonical  $\Phi \mathcal{A} \boxtimes \mathcal{A}^{op}$ -module structure on  $\mathcal{B}$ . Indeed, it suffices to construct commuting actions  $b(\Phi \mathcal{A}) \circ \mathcal{B} \rightarrow \mathcal{B}$  and  $\mathcal{B} \circ b(\mathcal{A}) \rightarrow \mathcal{B}$  compatible with  $\mathcal{O}_{X \times_S Y}$ -module structure. The right action of  $\mathcal{A}$  is obvious while the left action of  $\Phi \mathcal{A}$  on  $\mathcal{B}$  is given by the following map:

$$\Phi b(\mathcal{A}) \circ_{\mathcal{O}_X} \mathcal{B} = P \circ_{\mathcal{O}_Y} b(\mathcal{A}) \circ_{\mathcal{O}_Y} Q \circ_{\mathcal{O}_X} P \circ_{\mathcal{O}_Y} b(\mathcal{A}) \rightarrow P \circ_{\mathcal{O}_Y} b(\mathcal{A}) \circ_{\mathcal{O}_Y} b(\mathcal{A}) \rightarrow P \circ_{\mathcal{O}_Y} b(\mathcal{A}) = \mathcal{B},$$

where the last arrow is induced by multiplication in  $\mathcal{A}$ . It is clear that this is a morphism of right  $p_Y^{-1} \mathcal{A}$  modules. On the other hand, there is a natural isomorphism of sheaves on  $X \times_S Y$

$$b(\Phi \mathcal{A}) \circ_{\mathcal{O}_X} P \simeq P \circ_{\mathcal{O}_Y} b(\mathcal{A}) \circ_{\mathcal{O}_Y} Q \circ_{\mathcal{O}_X} P \simeq P \circ_{\mathcal{O}_Y} b(\mathcal{A}) \simeq \mathcal{B}.$$

One can easily check that the above left action of  $p_X^{-1} \Phi \mathcal{A}$  on  $\mathcal{B}$  is compatible with the natural left  $p_X^{-1} \Phi \mathcal{A}$ -module structure on  $b(\Phi \mathcal{A}) \circ_{\mathcal{O}_X} P$  via this isomorphism. Thus,  $\mathcal{B}$  is an object of  $\mathcal{D}^b(\mathcal{M}(\Phi \mathcal{A} \boxtimes \mathcal{A}^{op}))$  (concentrated in one degree). So we can define the functor

$$\Phi : \mathcal{D}^b \mathcal{M}(\mathcal{A}) \rightarrow \mathcal{D}^b \mathcal{M}(\Phi \mathcal{A}) : M \mapsto \mathcal{B} \circ_{\mathcal{A}} M$$

Similarly, we define an  $\mathcal{A} \boxtimes \Phi \mathcal{A}^{op}$ -module (perhaps shifted) on  $Y \times_S X$ :

$$\mathcal{B}' = b(\mathcal{A}) \circ_{\mathcal{O}_Y} Q \simeq Q \circ_{\mathcal{O}_X} b(\Phi \mathcal{A})$$

and the functor

$$\Phi' : \mathcal{D}^b \mathcal{M}(\Phi \mathcal{A}) \rightarrow \mathcal{D}^b \mathcal{M}(\mathcal{A}) : N \mapsto \mathcal{B}' \circ_{\Phi \mathcal{A}} N.$$

One has an isomorphism in the derived category of right  $\mathcal{O}_Y \boxtimes \mathcal{A}$ -modules on  $Y \times_S Y$

$$\mathcal{B}' \circ_{\Phi \mathcal{A}} \mathcal{B} \simeq (Q \circ_{\mathcal{O}_X} b(\Phi \mathcal{A})) \circ_{\Phi \mathcal{A}} \mathcal{B} \simeq Q \circ_{\mathcal{O}_X} \mathcal{B} \simeq Q \circ_{\mathcal{O}_X} P \circ_{\mathcal{O}_Y} b(\mathcal{A}) \simeq b(\mathcal{A}).$$

Similarly, there is an isomorphism in the derived category of left  $\mathcal{A} \boxtimes \mathcal{O}_Y$ -modules

$$\mathcal{B}' \circ_{\Phi \mathcal{A}} \mathcal{B} \simeq \mathcal{B}' \circ_{\Phi \mathcal{A}} (b(\Phi \mathcal{A}) \circ_{\mathcal{O}_X} P) \simeq \mathcal{B}' \circ_{\mathcal{O}_X} P \simeq b(\mathcal{A}) \circ_{\mathcal{O}_Y} Q \circ_{\mathcal{O}_X} P \simeq b(\mathcal{A}).$$

Moreover, both these isomorphisms coincide with the following isomorphism of  $\mathcal{O}_{Y \times_S Y}$ -modules

$$\mathcal{B}' \circ_{\Phi \mathcal{A}} \mathcal{B} \simeq Q \circ_{\mathcal{O}_X} b(\Phi \mathcal{A}) \circ_{\Phi \mathcal{A}} b(\Phi \mathcal{A}) \circ_{\mathcal{O}_X} P \simeq Q \circ_{\mathcal{O}_X} b(\Phi \mathcal{A}) \circ_{\mathcal{O}_X} P \simeq b(\mathcal{A}).$$

It follows that  $\mathcal{B}' \circ_{\Phi \mathcal{A}} \mathcal{B} \simeq b(\mathcal{A})$  in the derived category of  $\mathcal{A} \boxtimes \mathcal{A}^{op}$ -modules,

Similarly,  $\mathcal{B} \circ_{\mathcal{A}} \mathcal{B}' \simeq b(\Phi \mathcal{A})$ . It follows that the compositions  $\Phi \Phi' : \mathcal{D}^b \mathcal{M}(\Phi \mathcal{A}) \rightarrow \mathcal{D}^b \mathcal{M}(\Phi \mathcal{A})$  and  $\Phi' \Phi : \mathcal{D}^b \mathcal{M}(\mathcal{A}) \rightarrow \mathcal{D}^b \mathcal{M}(\mathcal{A})$  are identity functors.

The composition of  $\Phi$  with the forgetting functor can be easily computed:

$$\mathcal{B} \circ_{\mathcal{A}} M \simeq (P \circ_{\mathcal{O}_Y} b(\mathcal{A})) \circ_{\mathcal{A}} M \simeq P \circ_{\mathcal{O}_Y} M .$$

Hence, forgetting  $\Phi \mathcal{A}$ -module structure, we just get the transform with kernel  $P$ .  $\square$

*Remarks 2.1.* 1. In the situation of the theorem if we have another special  $D$ -algebra  $\mathcal{A}'$  and a homomorphism of  $D$ -algebras  $\mathcal{A} \rightarrow \mathcal{A}'$  then we have the corresponding induction and restriction functors  $M \mapsto \mathcal{A}' \otimes_{\mathcal{A}} M$  and  $N \mapsto N$  between categories of  $\mathcal{A}$ -modules and  $\mathcal{A}'$ -modules. It is easy to check that the corresponding derived functors commute with our functors  $\Phi$  constructed for  $\mathcal{A}$  and  $\mathcal{A}'$ .

2. Let  $A$  be an abelian variety,  $\hat{A}$  be the dual abelian variety. Then as was shown in [L] and [R2] the Fourier-Mukai equivalence  $\mathcal{D}^b(A) \simeq \mathcal{D}^b(\hat{A})$  extends to an equivalence of the derived categories of  $\mathcal{D}$ -modules on  $A$  and  $\mathcal{O}$ -modules on the universal extension of  $\hat{A}$  by a vector space. The latter category is equivalent to the category of modules over the commutative sheaf of algebras  $\mathcal{A}$  on  $\hat{A}$  which is constructed as follows. Let

$$0 \rightarrow \mathcal{O} \rightarrow \mathcal{E} \rightarrow H^1(\hat{A}, \mathcal{O}) \otimes \mathcal{O} \rightarrow 0$$

be the universal extension. Then  $\mathcal{A} = \text{Sym}(\mathcal{E})/(1_{\mathcal{E}} - 1)$  where  $1_{\mathcal{E}}$  is the image of  $1 \in \mathcal{O}$  in  $\mathcal{E}$ . It is easy to see that  $\mathcal{A}$  is the dual special  $D$ -algebra to the algebra of differential operators on  $A$ , so our theorem implies the mentioned equivalence of categories.

3. In the case of abelian varieties one can generalize the notion of special  $D$ -algebra as follows. Instead of considering special sheaves on  $X \times X$  one can consider quasi-coherent sheaves on  $X \times X$  admitting filtration with quotients of the form  $(\text{id}, t_x)_* L$  where  $(\text{id}, t_x) : X \rightarrow X \times X$  is the graph of the translation by some point  $x \in X$ ,  $L$  is a line bundle algebraically equivalent to zero on  $X$ . Let us call such sheaves quasi-special. It is easy to see that quasi-special sheaves are flat over  $X$  with respect to both projections  $p_1$  and  $p_2$ , so the operation  $\circ$  is exact on them. We can define a quasi-special algebra as a quasi-special sheaf  $K$  on  $X \times X$  together with the associative multiplication  $K \circ K \rightarrow K$  admitting a unit  $\Delta_* \mathcal{O}_X \rightarrow K$ . Then there is a Fourier duality for quasi-special algebras and equivalence of the corresponding derived categories. The proof of the above theorem works literally in this situation. Note that modules over quasi-special algebras form much broader class of categories than those over special  $D$ -algebras. Among these categories we can find some categories of modules over 1-motives and our Fourier duality coincides with the one defined by G. Laumon in [L]. For example, a homomorphism  $\phi : \mathbb{Z} \rightarrow X$  defines a quasi-special algebra on  $X$  which is a sum of structural sheaves of graphs of translations by  $\phi(n)$ ,  $n \in \mathbb{Z}$ . The corresponding category of modules is the category of  $\mathbb{Z}$ -equivariant  $\mathcal{O}_X$ -modules. The Fourier dual algebra corresponds to the affine group over  $\hat{X}$  which is an extension of  $\hat{X}$  by the multiplicative group.

**2.3.** Let  $L$  be a Lie algebroid on  $Y$  such that  $L \simeq \mathcal{O}_Y^d$  as an  $\mathcal{O}_Y$ -module. Then for any central extension  $\tilde{L}$  of  $L$  by  $\mathcal{O}_Y$ , the  $D$ -algebra  $\mathcal{U}^\circ(\tilde{L})$  is special. Furthermore, one has  $\Phi \mathcal{U}^\circ(\tilde{L}) \simeq \mathcal{U}^\circ(\tilde{L}')$  for some central extension  $\tilde{L}'$  of a Lie algebroid  $L'$  on  $X$  such that  $L' \simeq \mathcal{O}_X^d$  as an  $\mathcal{O}_X$ -module. Indeed, this follows essentially from



Lemma 1.1. One just has to notice that if a  $D$ -algebra  $\mathcal{A}$  on  $Y$  has an algebra filtration  $\mathcal{A}_\bullet$  with  $\text{gr } \mathcal{A}_\bullet \simeq S^\bullet(\mathcal{O}_Y^d)$ , then  $\Phi\mathcal{A}$  has an algebra filtration  $\mathcal{F}\mathcal{A}_\bullet$  with  $\text{gr } \Phi\mathcal{A}_\bullet \simeq S^\bullet(\mathcal{O}_X^d)$ . Note that if  $L$  is a successive extension of trivial bundles then the  $D$ -algebra  $\mathcal{U}^\circ(\tilde{L})$  is still special, but  $\Phi\mathcal{U}^\circ(\tilde{L})$  is not necessarily of the form  $\mathcal{U}^\circ(\tilde{L}')$ .

**2.4.** Assume now that  $X$  is an abelian variety. Let  $\mathcal{P}$  be a Picard algebroid on  $\hat{X}$ ,  $\mathcal{D} = \mathcal{U}^\circ(\mathcal{P})$  be the corresponding  $tdo$ . Then  $\mathcal{P}/\mathcal{O}_{\hat{X}} \simeq \mathcal{T}_{\hat{X}} \simeq \hat{\mathfrak{g}} \otimes_k \mathcal{O}_{\hat{X}}$  is a trivial  $\mathcal{O}_{\hat{X}}$ -module, hence  $\Phi\mathcal{D} \simeq \mathcal{U}^\circ(\tilde{L}')$  for some Lie algebroid  $L'$  on  $X$  and its central extension  $\tilde{L}'$  by  $\mathcal{O}_X$ .

**Proposition 2.4.** *Let  $\mathcal{D}$  be a  $tdo$  on  $\hat{X}$ ,  $\mathcal{P}$  be the corresponding Picard algebroid. Then  $\Phi\mathcal{D}$  is a  $tdo$  on  $X$  if and only if the map  $\hat{\mathfrak{g}} \rightarrow H^1(\hat{X}, \mathcal{O})$ , induced by the extension of  $\mathcal{O}_{\hat{X}}$ -modules*

$$0 \rightarrow \mathcal{O}_{\hat{X}} \rightarrow \mathcal{P} \rightarrow \hat{\mathfrak{g}} \otimes_k \mathcal{O}_{\hat{X}} \rightarrow 0,$$

*is an isomorphism.*

*Proof.* Let  $\mathcal{D}_\bullet$  be the canonical filtration of  $\mathcal{D}$ . Then  $\Phi\mathcal{D}$  is a  $tdo$  if only if the class of the extension of  $\mathcal{O}_X$ -bimodules

$$0 \rightarrow \mathcal{O}_X \simeq \Phi\mathcal{D}_0 \rightarrow \Phi\mathcal{D}_1 \rightarrow \Phi(\mathcal{D}_1/\mathcal{D}_0) \simeq \hat{\mathfrak{g}} \otimes \mathcal{O}_X \rightarrow 0$$

induces an isomorphism  $\hat{\mathfrak{g}} \otimes \mathcal{O}_{\hat{X}} \rightarrow \mathcal{T}_{\hat{X}}$ . Thus, it is sufficient to check that the components of the canonical decomposition

$$\text{Ext}_{\mathcal{O}_{X \times X}}^1(\Delta_* \mathcal{O}_X, \Delta_* \mathcal{O}_X) \simeq H^0(X, \mathcal{T}) \oplus H^1(X, \mathcal{O}_X)$$

introduced in Lemma 1.2, get interchanged by the Fourier-Mukai transform, if we take into account the natural isomorphisms

$$H^0(X, \mathcal{T}) \simeq \mathfrak{g} \simeq H^1(\hat{X}, \mathcal{O}),$$

$$H^1(X, \mathcal{O}_X) \simeq \hat{\mathfrak{g}} \simeq H^0(\hat{X}, \mathcal{T}).$$

We leave this to the reader as a pleasant exercise on Fourier-Mukai transform.  $\square$

**2.5.** Let us describe in more details the data describing a Lie algebroid  $L$  on an abelian variety  $X$  such that  $L \simeq V \otimes_k \mathcal{O}_X$  as  $\mathcal{O}_X$ -module, where  $V$  is a finite-dimensional  $k$ -vector space, and a central extension  $\tilde{L}$  of  $L$  by  $\mathcal{O}_X$ . First of all,  $V = H^0(X, L)$  has a structure of Lie algebra, and the structural morphism  $L \rightarrow \mathcal{T}$  is given by some  $k$ -linear map  $\beta : V \rightarrow \mathfrak{g} = H^0(X, \mathcal{T})$  which is a homomorphism of Lie algebras (where  $\mathfrak{g}$  is an abelian Lie algebra). The central extension  $\tilde{L}$  is described (up to an isomorphism) by a class  $\tilde{\alpha}$  in the first hypercohomology space  $\mathbb{H}^1(X, L^* \rightarrow \wedge^2 L^* \rightarrow \wedge^3 L^* \rightarrow \dots)$  of the truncated Koszul complex of  $L$ . In particular, we have the corresponding class  $\alpha \in H^1(X, L^*)$ , which is just the class of the extension of  $\mathcal{O}_X$ -modules

$$0 \rightarrow \mathcal{O}_X \rightarrow \tilde{L} \rightarrow L \rightarrow 0.$$

We can consider  $\alpha$  as a linear map  $V \rightarrow H^1(X, \mathcal{O}_X) = \hat{\mathfrak{g}}$ . The maps  $\alpha$  and  $\beta$  get interchanged by the Fourier transform, up to a sign.

By definition the  $D$ -algebra associated with  $\tilde{L}$  is a tdo if and only if  $\beta : V \rightarrow \mathfrak{g}$  is an isomorphism. If an addition  $\alpha : V \rightarrow \hat{\mathfrak{g}}$  is an isomorphism then the dual  $D$ -algebra is also a tdo. Thus, we have a bijection between tdo's with non-degenerate first Chern class on  $X$  and  $\hat{X}$  such that the corresponding derived categories of modules are equivalent. According to [BB] isomorphism classes of tdo on  $X$  are classified by  $\mathbb{H}^2(X, \Omega^{\geq 1})$  which is an extension of  $H^1(X, \Omega^1) \simeq \text{Hom}(\mathfrak{g}, \hat{\mathfrak{g}})$  by  $H^0(X, \Omega^2) = \wedge^2 \mathfrak{g}^*$ . Let  $U_X \subset \mathbb{H}^2(X, \Omega^{\geq 1})$  be the subset of elements with non-degenerate projection to  $H^1(X, \Omega^1)$ . The duality gives an isomorphism between  $U_X$  and  $U_{\hat{X}}$ . It is easy to see that under this isomorphism multiplication by  $\lambda \in k^*$  on  $U_X$  corresponds to multiplication by  $\lambda^{-1}$  on  $U_{\hat{X}}$ .

On the other hand, let  $\mathcal{A}$  be a tdo with trivial  $c_1$ . In other words,  $\mathcal{A}$  corresponds to some global 2-form  $\omega$  on  $X$ . Modules over  $\mathcal{A}$  are  $\mathcal{O}$ -modules equipped with a connection having the scalar curvature  $\omega$ . Let  $\mathcal{B}$  be the dual  $D$ -algebra on  $\hat{X}$  and let  $\tilde{L} \rightarrow L = H^0(X, T_X) \otimes \mathcal{O}_{\hat{X}}$  be the corresponding central extension of Lie algebroids. We claim that  $L$  is just an  $\mathcal{O}_{\hat{X}}$ -linear commutative Lie algebra while the central extension  $\tilde{L}$  is given by the class  $(e, \omega) \in H^1(L^*) \oplus H^0(\wedge^2 L^*)$ , where  $e$  is the canonical element in  $H^1(L^*) \simeq H^1(\hat{X}, \mathcal{O}) \otimes H^1(\hat{X}, \mathcal{O})^*$ . Indeed, as an  $\mathcal{O}_{\hat{X}}$ -module  $\tilde{L}$  is a universal extension of  $H^1(\hat{X}, \mathcal{O}) \otimes \mathcal{O}$  by  $\mathcal{O}$ . Hence, the Lie bracket defines a morphism of  $\mathcal{O}$ -modules  $\wedge^2 L \rightarrow \tilde{L}$ . Since  $H^0(\tilde{L}) = H^0(\mathcal{O})$  it follows that  $[\tilde{L}, \tilde{L}] \subset \mathcal{O} \subset \tilde{L}$ . It is easy to see that the Lie bracket is just given by  $\omega : \wedge^2 L \rightarrow \mathcal{O}$ .

Recall that the Neron-Severi group of  $X$  is identified with  $\text{Hom}^{\text{sym}}(X, \hat{X}) \otimes \mathbb{Q}$  where  $\text{Hom}^{\text{sym}}(X, \hat{X})$  is the group of symmetric homomorphisms  $X \rightarrow \hat{X}$ . Namely, to a line bundle  $L$  there corresponds a symmetric homomorphism  $\phi_L : X \rightarrow \hat{X}$  sending a point  $x$  to  $t_x^* L \otimes L^{-1}$  where  $t_x : X \rightarrow X$  is the translation by  $x$ . One has the natural homomorphism  $c_1 : NS(X) \rightarrow \mathbb{H}^2(X, \Omega^{\geq 1})$  sending a line bundle  $L$  to the class of the ring  $D_L$  of differential operators on  $L$ . For  $\mu \in NS(X)$  we denote by  $D_\mu$  the corresponding tdo. If  $\mu \in NS(X)$  is a non-degenerate class so that  $c_1(\mu) \in U_X$  then the Fourier dual tdo to  $D_\mu$  is

$$(2.5.1) \quad \Phi(D_\mu) = D_{-\mu^{-1}}$$

Indeed, it suffices to check this when  $\mu$  is a class of a line bundle  $L$ , in which case it follows easily from the isomorphism

$$\phi_L^* \det \Phi(L) \simeq L^{-\text{rk } \Phi(L)}$$

and the fact that the dual tdo to  $D_L$  acts on  $\Phi(L)$ .

Let  $E$  be a coherent sheaf which is a module over some tdo on  $X$  (then  $E$  is automatically locally free). Following [BB] we say in this case that there is an integrable projective connection on  $E$ .

**Proposition 2.5.** *Let  $E$  be a vector bundle on  $X$  equipped with an integrable projective connection. Assume that  $\det E$  is a non-degenerate line bundle. Then  $H^i \Phi(E)$  are vector bundles with canonical integrable projective connections, and the following equality holds:*

$$\phi_{\det E}^* c_1(\Phi(E)) = -\chi(X, E) \cdot \text{rk } E \cdot c_1(E).$$

*Proof.* The first statement follows immediately from the fact that  $\Phi(E)$  is quasi-isomorphic to a complex of modules over the tdo on  $\hat{X}$  dual to  $D_{(\det E)^{\frac{1}{r}}}$  where

$r = \text{rk } E$ . On the other hand, this tdo acting on  $\Phi(E)$  is isomorphic  $D_{(\det \Phi(E))^{\frac{1}{r}}}$  where  $r' = \text{rk } \Phi(E) = \chi(X, E)$ . Considering classes of these dual tdo's and using the isomorphism (2.5.1) applied to  $\mu = \frac{1}{r} \phi_{\det E}$  we get the above formula.  $\square$

**2.6.** The following two natural questions arise: 1) whether for every  $\mu \in NS(X)$  there exists a vector bundle  $E$  on  $X$  which is a module over  $D_\mu$ , 2) what vector bundles on an abelian variety admit integrable projective connections. To answer these questions we use the following construction. Let  $\pi : X_1 \rightarrow X_2$  be an isogeny of abelian varieties and  $E$  be a vector bundle with an integrable projective connection on  $X_1$ . Then there is a canonical integrable projective connection on  $\pi_* E$ . Indeed, the simplest way to see this is to use Fourier duality. If  $E$  is a module over some tdo  $D_\lambda$  on  $X_1$  then  $\Phi(E)$  is a module over the dual  $D$ -algebra  $\Phi(D_\lambda)$  on  $\hat{X}_1$ . Now we use the formula

$$\pi_* E \simeq \Phi^{-1} \hat{\pi}^* (\Phi(E)),$$

where  $\Phi^{-1}$  is the inverse Fourier transform on  $X_2$ , hence  $\pi_* E$  is a module over  $\Phi^{-1} \hat{\pi}^* \Phi(D_\lambda)$  which is a tdo on  $X_2$ .

In particular, the push-forwards of line bundles under isogenies have canonical integrable projective connections. Also it is clear that if  $E$  is a vector bundle with an integrable projective connection and  $F$  is a flat vector bundle then  $E \otimes F$  has a natural integrable projective connection.

Now we can answer the above questions.

**Theorem 2.2.** *For every  $\mu \in NS(X)$  there exists a vector bundle  $E$  which is a module over  $D_\mu$ .*

*Proof.* We can write  $\mu = [L]/n$  where  $n > 0$  is an integer,  $[L]$  is a class of a line bundle  $L$  on  $X$ . Let  $[n]_A : A \rightarrow A$  be an endomorphism of multiplication by  $n$ . Then  $[n]_A^*(\mu) \in NS(X)$  is represented by a line bundle  $L'$ . Now we claim that the push-forward  $[n]_{A,*} L'$  has a structure of a module over  $D_\mu$ . Indeed, it suffices to check that

$$c_1([n]_{A,*} L') / \deg([n]_A) = \mu.$$

Let  $\text{Nm}_n : NS(X) \rightarrow NS(X)$  be the norm homomorphism corresponding to the isogeny  $[n]_A$ . Then the LHS of the above equality is  $\text{Nm}_n([L']) / \deg([n]_A)$ . Hence, the pull-back of the LHS by  $[n]_A$  is equal to  $[L'] = [n]_A^*(\mu)$  which implies our claim.  $\square$

**Theorem 2.3.** *Let  $E$  be an indecomposable vector bundle with an integrable projective connection on an abelian variety  $X$ . Then there exists an isogeny of abelian varieties  $\pi : X' \rightarrow X$ , a line bundle  $L$  on  $X'$  and a flat bundle  $F$  on  $X$  such that  $E \simeq \pi_* L \otimes F$ .*

*Proof.* The main idea is to analyze the sheaf of algebras  $A = \text{End}(E)$ . Namely,  $A$  has a flat connection such that the multiplication is covariantly constant. In other words, it corresponds to a representation of the fundamental group  $\pi_1(X)$  in automorphisms of the matrix algebra. Since all such automorphisms are inner we get a homomorphism

$$\rho : \pi_1(X) \rightarrow PGL(E_0)$$

where  $E_0$  is a fiber of  $E$  at zero. Now the central extension  $SL(E_0) \rightarrow PGL(E_0)$  induces a central extension of  $\pi_1(X) = \mathbb{Z}^{2g}$  by the group of roots of unity of order

$rkE$ . This central extension splits on some subgroup of finite index  $H \subset \pi_1(X)$ . In other words, the restriction of  $\rho$  to  $H$  lifts to a homomorphism  $\rho_H : H \rightarrow GL(E_0)$ . Let  $\pi : \tilde{X} \rightarrow X$  be an isogeny corresponding to  $H$ , so that  $\tilde{X}$  is an abelian variety with  $\pi_1(\tilde{X}) = H$ . Then  $\rho_H$  defines a flat bundle  $\tilde{F}$  on  $\tilde{X}$  such that

$$\pi^* A \simeq \text{End}(\tilde{F})$$

as algebras with connections. It follows that  $\pi^* E \simeq L \otimes \tilde{F}$  for some line bundle  $L$  on  $\tilde{X}$ . Thus,  $L$  is a direct summand of  $\pi_*(L \otimes \tilde{F})$ . Note that there exists a flat bundle  $F$  on  $X$  such that  $\tilde{F} \simeq \pi^* F$  (again the simplest way to see this is to use the Fourier duality). Hence,  $L$  is a direct summand of  $\pi_* L \otimes F$ . It remains to check that all indecomposable summands of the latter bundle have the same form. This follows from the following lemma.  $\square$

**Lemma 2.2.** *Let  $\pi : X_1 \rightarrow X_2$  be an isogeny of abelian varieties,  $L$  be a line bundle on  $X_1$ ,  $F$  be an indecomposable flat bundle on  $X_1$ . Assume that  $\pi_*(L \otimes F)$  is decomposable. Then there exists a non-trivial factorization of  $\pi$  into a composition*

$$X_1 \xrightarrow{\pi'} X'_1 \rightarrow X_2$$

*such that  $L \simeq (\pi')^* L'$  for some line bundle  $L'$  on  $X'_1$ .*

*Proof.* By adjunction and projection formula we have

$$\text{End}(\pi_*(L \otimes F)) \simeq \text{Hom}(\pi^* \pi_*(L \otimes F), L \otimes F) \simeq \bigoplus_{x \in K} \text{Hom}(t_x^* L \otimes F, L \otimes F)$$

where  $K \subset X_1$  is the kernel of  $\pi$ . If  $t_x^* L \simeq L$  for some  $x \in K$ ,  $x \neq 0$  then  $L$  descends to a line bundle on the quotient of  $X_1$  by the subgroup generated by  $x$ . Otherwise, we get  $\text{End}(\pi_*(L \otimes F)) \simeq \text{End}(F)$ , hence,  $\pi_*(L \otimes F)$  is indecomposable.  $\square$

### 3. NONCOMMUTATIVE ÉTALE MORPHISMS

This section provides the setting we will need to discuss microlocalization.

#### 3.1.

**Definition 3.1.** (cf. [K]) A ring homomorphism  $A' \xrightarrow{\phi} A$  is called a central extension if  $\phi$  is surjective,  $\ker(\phi)$  is a central ideal and  $\ker(\phi)^2 = 0$ .

**Definition 3.2.** Let **Rings** denote the category of associative rings. Let  $\mathcal{C} \subset \mathbf{Rings}$  be a full subcategory. A morphism  $R \xrightarrow{\alpha} S$  in  $\mathcal{C}$  is formally étale if, for every commutative diagram  $\alpha, \beta, \gamma, \delta$

$$(3.1.1) \quad \begin{array}{ccc} R & \xrightarrow{\delta} & A' \\ \alpha \downarrow & \nearrow \epsilon & \downarrow \gamma \\ S & \xrightarrow{\beta} & A \end{array}$$

in  $\mathcal{C}$ , with  $\gamma$  a central extension, there exists a unique morphism  $S \xrightarrow{\epsilon} A'$  such that diagram (3.1.1) commutes.

**Example 3.1.** Let  $R$  be a ring and let  $a_0, a_1, \dots, a_n \in R$ . Let  $S$  be the  $R$ -algebra generated by elements  $z, u$  subject to the relations

$$(3.1.2) \quad \sum a_i z^i = 0, \quad u \sum i a_i z^{i-1} = 1, \quad \sum i a_i z^{i-1} u = 1.$$

Then the natural map  $R \xrightarrow{\alpha} S$  is formally étale in **Rings**.

*Proof.* Consider a commutative diagram

$$(3.1.3) \quad \begin{array}{ccc} R & \xrightarrow{\delta} & A' \\ \alpha \downarrow & & \downarrow \gamma \\ S & \xrightarrow{\beta} & A \end{array}$$

as in definition 3.2. Let  $I = \ker(\gamma)$ . Choose  $x, y \in A'$  such that  $\gamma(x) = \beta(z)$ ,  $\gamma(y) = \beta(u)$ . It must be shown that there are unique elements  $p, q \in I$  such that

$$(3.1.4) \quad \begin{aligned} \sum \delta(a_i)(x+p)^i &= 0, \\ (y+q) \sum i \delta(a_i)(x+p)^{i-1} &= 1, \\ \sum i \delta(a_i)(x+p)^{i-1}(y+q) &= 1. \end{aligned}$$

Since  $I^2 = 0$ , the equations are uniquely solved by setting

$$(3.1.5) \quad p = -y \sum \delta(a_i) x^i, \quad q = y(1 - \sum i \delta(a_i)(x+p)^{i-1} y).$$

□

**3.2.** Let us recall some definitions from [K]. For any associative algebra  $R$ , the NC-filtration on  $R$  is the decreasing filtration  $\{F^d R\}_{d \geq 0}$  defined by setting

$$F^d R = \sum_{i_1 + \dots + i_m = d} R \cdot R_{i_1} \cdot R \cdot \dots \cdot R \cdot R_{i_m} \cdot R$$

where  $R_0 = R$ ,  $R_{i+1} = [R, R_i]$  are the terms of the lower central series for  $R$  considered as a Lie algebra (we use a different indexing from Kapranov's). This filtration is compatible with multiplication and the associated graded algebra is commutative.

The category  $\mathcal{N}_d$  is the category of associative algebras  $R$  with  $F^{d+1} R = 0$ . For example,  $\mathcal{N}_0$  is the category of commutative algebras. For every  $d$  there is a pair of adjoint functors  $r_d : \mathcal{N}_d \rightarrow \mathcal{N}_{d-1}$  and  $i_d : \mathcal{N}_{d-1} \rightarrow \mathcal{N}_d$ , where  $i_d$  is the natural inclusion,  $r_d(R) = R/F_d R$ . Note that if  $R \in \mathcal{N}_d$  then  $F^d R \subset R$  is a central ideal with zero square. Thus,  $R$  is a central extension of  $r_d(R)$ . Indeed,  $\mathcal{N}_d$  is the category of rings  $A$  which are obtained as the composition of  $d$  central extensions,

$$A \rightarrow A_1 \rightarrow A_2 \rightarrow \dots \rightarrow A_d$$

with  $A_d$  commutative.

**Lemma 3.1.** *Let  $R \rightarrow S$  be a formally étale morphism in  $\mathcal{N}_d$ ,  $M$  be an  $S^{ab}$ -module. Then the natural map  $\text{Der}(S, M) \rightarrow \text{Der}(R, M)$  is a bijection.*

*Proof.* Given a central  $S$ -bimodule  $M$  we can define a trivial central extension of  $S$  by  $M$ :  $\tilde{S} = S \oplus M$ . Then derivations from  $S$  to  $M$  are in bijective correspondence with splittings of the projection  $\tilde{S} \rightarrow S$ . Hence, the assertion.  $\square$

**Proposition 3.1.** *Let  $R \xrightarrow{\alpha} S$  be a formally étale morphism in  $\mathcal{N}_d$ . Let*

$$(3.2.1) \quad \begin{array}{ccc} R & \xrightarrow{\delta} & A' \\ \downarrow \alpha & & \downarrow \gamma \\ S & \xrightarrow{\beta} & A \end{array}$$

*be a commutative diagram in **Rings**, such that  $\beta$  is surjective and  $\gamma$  is a central extension. Then  $A' \in \mathcal{N}_d$ .*

*Proof.* Note that apriori we know from this diagram that  $A \in \mathcal{N}_d$ , hence,  $A' \in \mathcal{N}_{d+1}$ . We want to prove that  $F^{d+1}A' = 0$ . Since  $F^{d+2}A' = 0$  it suffices to prove that for every sequence of positive numbers  $i_1, \dots, i_m$  such that  $i_1 + \dots + i_m = d + 1$  one has

$$A'_{i_1} \cdot A'_{i_2} \cdot \dots \cdot A'_{i_m} = 0.$$

We use descending induction in  $m$ . Assume that this is true for  $m + 1$ . Then we can define a map

$$D : S^{m+d+1} \rightarrow I :$$

$$(s_1, \dots, s_{m+d+1}) \mapsto [a'_1, [\dots, [a'_{i_1}, a'_{i_1+1}] \dots]] \cdot [a'_{i_1+2}, [\dots, [a'_{i_1+i_2+1}, a'_{i_1+i_2+2}] \dots]] \cdot \dots$$

where  $a'_i \in A'$  are such that  $\gamma(a'_i) = \beta(s_i)$ . This map is well-defined since  $a'_i$  are well-defined modulo  $I$  which is a central ideal. Now the induction assumption implies that  $D$  is a derivation in every argument. Hence, applying Lemma 3.1 we conclude that  $D = 0$ .  $\square$

**Theorem 3.1.** *Let  $R \xrightarrow{\alpha} S$  be a formally étale morphism in  $\mathcal{N}_d$ . Then  $R \xrightarrow{\alpha} S$  is a formally étale morphism in **Rings**.*

*Proof.* This follows easily from proposition 3.1.  $\square$

Let  $\mathcal{N}_\infty$  denote the category of rings that are complete with respect to the NC-filtration.

**Theorem 3.2.** *Let  $R \xrightarrow{\alpha} S$  be formally étale in  $\mathcal{N}_\infty$ , with  $R \in \mathcal{N}_d$ . Then  $S \in \mathcal{N}_d$ .*

*Proof.* The natural morphism  $R \rightarrow r_{d+i}(S)$  is formally étale in  $\mathcal{N}_{d+i}$  for all  $i \geq 0$ . Proposition 3.1 applied to the diagram

$$(3.2.2) \quad \begin{array}{ccc} R & \xrightarrow{\quad} & r_{d+i+1}(S) \\ \downarrow & & \downarrow \\ r_{d+i}(S) & \xrightarrow{=} & r_{d+i}(S) \end{array}$$

shows that  $F^{d+i}(S) \subset F^{d+i+1}(S)$  for all  $i \geq 1$ . Hence the assertion.  $\square$

**Example 3.2.** Let  $R$  be a ring belonging  $\mathcal{N}_d$  for some  $d$ . Let  $R \xrightarrow{\alpha} S$  be as in example 3.1. Let  $\hat{S}$  denote the completion of  $S$  with respect to the NC-filtration. Then  $\hat{S}$  belongs to  $\mathcal{N}_d$  and the natural morphism  $R \rightarrow \hat{S}$  is formally étale (in **Rings**). As in the commutative case, we call such a morphism *standard*.

**3.3.** The category  $\text{NCS}^d$  of NC-schemes of degree  $d$  (in Kapranov's terminology "NC-nilpotent of degree  $d$ ") is constructed in the same way as the commutative category of schemes using  $\mathcal{N}_d$  instead of  $\mathcal{N}_0$  as coordinate rings of affine schemes. For a scheme  $X \in \text{NCS}^d$  we denote by  $\text{NCS}_{/X}^d$  the category of NC-schemes of degree  $d$  over  $X$ . The morphisms in  $\text{NCS}_X^d$  are denoted by  $\text{Hom}_X(\cdot, \cdot)$ .

As in affine case we have natural adjoint functors  $r_d : \text{NCS}^d \rightarrow \text{NCS}^{d-1}$  and  $i_d : \text{NCS}^{d-1} \rightarrow \text{NCS}^d$ . In particular, we have the abelianization functor  $\text{NCS}^d \rightarrow \text{NCS}^0 : X \mapsto X^{ab}$  given by the composition  $r_1 r_2 \dots r_d$ .

A morphism  $Z \rightarrow \tilde{Z}$  of NC-schemes of degree  $d$  is called a nilpotent thickening if it induces an isomorphism of underlying topological spaces and  $\mathcal{O}_{\tilde{Z}} \rightarrow \mathcal{O}_Z$  is a surjection with nilpotent kernel.

**Definition 3.3.** A morphism  $Y \rightarrow X$  in  $\text{NCS}^d$  is called formally smooth (resp. formally unramified) if for every nilpotent thickening  $Z \subset \tilde{Z}$  in  $\text{NCS}_{/X}^d$  the map  $\text{Hom}_X(\tilde{Z}, Y) \rightarrow \text{Hom}_X(Z, Y)$  is surjective (resp. injective). A morphism is called formally étale if it is both formally smooth and formally unramified. A morphism is called étale if it is étale and the corresponding morphism of commutative schemes  $Y^{ab} \rightarrow X^{ab}$  is locally of finite type.

**Proposition 3.2.** Let  $P$  be a property of being formally smooth (resp. formally unramified, resp. formally étale).

- a) Let  $f : Y \rightarrow X$  be a morphism in  $\text{NCS}^d$  with property  $P$ . Then the same property holds for  $r_d(f)$ .
- b) If  $f : Y \rightarrow X$  and  $g : Z \rightarrow Y$  are morphisms in  $\text{NCS}^d$  having property  $P$  then  $f \circ g$  also has this property.
- c) If  $f : Y \rightarrow X$  is formally unramified morphism in  $\text{NCS}^d$ ,  $g : Z \rightarrow Y$  is a morphism in  $\text{NCS}^d$  such that  $f \circ g$  has property  $P$  then  $g$  also has this property.
- d) An open morphism  $U \rightarrow X$  is étale.
- e) A morphism  $f : Y \rightarrow X$  is étale if and only if there exists an open covering  $X = \cup X_i$  and for every  $i$  an open covering  $Y_{ij}$  of  $f^{-1}(X_i)$  such that all the induced morphisms  $Y_{ij} \rightarrow X_i$  are étale.

The proof is straightforward.

Theorem 3.1 has the following global version.

**Theorem 3.3.** Let  $f$  be a formally étale morphism in  $\text{NCS}^{d-1}$ . Then  $i_d(f)$  is a formally étale morphism in  $\text{NCS}^d$ .

Now we observe that the topological invariance of étale morphisms remains valid in the present context.

**Theorem 3.4.** For any  $X \in \text{NCS}^d$  the canonical functor  $Y \mapsto Y^{ab}$  from the category of étale  $X$ -schemes to that of étale  $X^{ab}$ -schemes is an equivalence.

*Proof.* First we claim that the functor in question is fully faithful. Indeed, let  $Y_1, Y_2 \in \text{NCS}^d$  be étale  $X$ -schemes. Then since  $Y_1^{ab} \rightarrow Y_1$  is a nilpotent thickening and  $Y_2 \rightarrow X$  is étale the natural map

$$\text{Hom}_X(Y_1, Y_2) \rightarrow \text{Hom}_X(Y_1^{ab}, Y_2) \simeq \text{Hom}_{X^{ab}}(Y_1^{ab}, Y_2^{ab})$$

is an isomorphism as required.

To prove the surjectivity of the functor it suffices to do it locally. Thus we may assume that the morphism  $Y^{ab} \rightarrow X^{ab}$  is a standard étale extension of commutative rings  $R^{ab} \rightarrow S^{ab}$  where  $S^{ab} = (R^{ab}[z_0]/(f_0(z_0)))_{f_0'(z_0)}$ ,  $f_0$  is a unital polynomial. But such a morphism lifts to a standard étale morphism in  $\mathcal{N}_d$ , as in example 3.2.  $\square$

**Corollary 3.1.** *A morphism  $f : Y \rightarrow X$  in  $\text{NCS}^d$  is étale if and only if there exists an open covering  $Y = \cup Y_i$  such that all the induced morphisms  $Y_i \rightarrow f(Y_i)$  are standard étale morphisms.*

#### 4. MICROLOCALIZATION

**4.1.** Let us return to the setting of Theorem (2.1). We assume that the  $D$ -algebra  $\mathcal{A}$  is equipped with an increasing algebra filtration  $\mathcal{A}_\bullet$  (so  $\mathcal{A}_i \mathcal{A}_j \subset \mathcal{A}_{i+j}$ ) such that  $\mathcal{A}_{-1} = 0$ ,  $\mathcal{A}_0 = \mathcal{O}_Y$ , the associated graded algebra  $\text{gr}(\mathcal{A})$  is commutative and is generated by  $\text{gr}(\mathcal{A})_1$  over  $\mathcal{O}_Y$ . In particular, the left and right actions of  $\mathcal{O}_Y$  on  $\text{gr}(\mathcal{A})_i$  are the same. We will call such a filtration *special* if there exists a sheaf of flat, commutative graded  $\mathcal{O}_S$ -algebras  $C$ , generated over  $\mathcal{O}_S$  by  $C_1$ , and an isomorphism of graded algebras  $\text{gr}(\mathcal{A}) \simeq \pi_S^{Y*}(C)$ . By lemma 2.1, such an isomorphism induces an isomorphism  $\text{gr}(\Phi\mathcal{A}) \simeq \pi_S^{Y*}(C)$ .

Given a  $D$ -algebra  $\mathcal{A}$  with a special filtration  $\mathcal{A}_\bullet$  we can form the corresponding sequence of graded algebras  $\text{gr}_{(n)}(\mathcal{A})$  for  $n \geq 0$  by setting

$$\text{gr}_{(n)}(\mathcal{A}) = \oplus_{i=0}^{\infty} \mathcal{A}_i / \mathcal{A}_{i-n-1}.$$

In particular,  $\text{gr}_{(0)}(\mathcal{A}) = \text{gr}(\mathcal{A})$  is commutative while for  $n \geq 1$  there is a central element  $t$  in  $\text{gr}_{(n)}(\mathcal{A})_1 = \mathcal{A}_1$  (corresponding to  $1 \in \mathcal{A}_0 \subset \mathcal{A}_1$ ) such that  $t^{n+1} = 0$  and  $\text{gr}_{(n)}(\mathcal{A})/(t) = \text{gr}(\mathcal{A})$ . These algebras form a projective system via the natural projections  $\text{gr}_{(n+1)}(\mathcal{A}) \rightarrow \text{gr}_{(n)}(\mathcal{A})$ .

Consider for each  $n$  the NC-scheme  $\mathbb{P}_n(\mathcal{A}) = \text{Proj}(\text{gr}_{(n)}(\mathcal{A}))$  corresponding to  $\text{gr}_{(n)}(\mathcal{A})$  via the noncommutative analogue of Proj construction. We denote by  $\mathcal{D}^-(\mathbb{P}_n(\mathcal{A}))$  the (bounded from above) derived category of left quasi-coherent sheaves  $\mathbb{P}_n(\mathcal{A})$ . Similar to the commutative case there is a natural localization functor  $M \mapsto \widetilde{M}$  from the category of graded  $\text{gr}_{(n)}(\mathcal{A})$ -modules to the category of quasi-coherent sheaves on  $\mathbb{P}_n(\mathcal{A})$ .

If  $M$  is a left  $\mathcal{A}$ -module (quasi-coherent over  $\mathcal{O}_Y$ ) equipped with an increasing module filtration  $M_\bullet$  then for every  $n > 0$  we can form the corresponding graded  $\text{gr}_{(n)}(\mathcal{A})$ -module  $\oplus M_i / M_{i-n}$ , hence the corresponding quasi-coherent sheaf

$$\text{ml}_n(M) = (\oplus_i M_i / M_{i-n})^\sim.$$

The above NC-schemes are connected by a sequence of closed embeddings  $i_n : \mathbb{P}_n(\mathcal{A}) \hookrightarrow \mathbb{P}_{n+1}(\mathcal{A})$  and the quasi-coherent sheaves  $\text{ml}_n(M)$  satisfy  $\text{ml}_n(M) =$



$\mathrm{ml}_{n+1}(M)|_{\mathbb{P}_n(\mathcal{A})}$ . In other words, the system  $(\mathrm{ml}_n(M))$  corresponds to a quasi-coherent sheaf on the formal NC-scheme  $\mathbb{P}_\infty(\mathcal{A}) = \mathrm{inj.} \lim \mathbb{P}_n(\mathcal{A})$ .

Our aim now is to establish an equivalence of derived categories of sheaves on  $\mathbb{P}_n(\mathcal{A})$  and  $\mathbb{P}_n(\Phi\mathcal{A})$ . We will prove something stronger – namely that such an equivalence exists étale locally on  $\mathrm{Proj}(C)$ .

With begin a Zariski local version. Clearly,  $\circ$  commutes with flat base change on  $S$ , so we may assume  $S$  is affine. The isomorphisms  $\mathrm{gr}(\mathcal{A}) \simeq \pi_S^{Y*}(\mathcal{C})$  and  $\mathrm{gr}(\Phi\mathcal{A}) \simeq \pi_S^{X*}(\mathcal{C})$  give us isomorphisms  $\mathbb{P}_1(\mathcal{A}) = Y \times_S \mathrm{Proj}(C)$  and  $\mathbb{P}_1(\Phi\mathcal{A}) = X \times_S \mathrm{Proj}(C)$ . Let  $f$  be a section of  $C_1$ . It defines a Zariski open subset  $D_f \subset \mathrm{Proj}(C)$  which is the spectrum of  $C_{(f)}$ , the degree zero part in the localization of  $C$  by  $f$ . Hence, we have the corresponding open subset  $Y \times_S D_f \subset \mathbb{P}_0(\mathcal{A})$ . Since  $\mathbb{P}_n$  has the same underlying topological space as  $\mathbb{P}_0$  we have the corresponding open subscheme  $\mathbb{P}_n(\mathcal{A})_f \subset \mathbb{P}_n(\mathcal{A})$  for every  $n \geq 0$ . We claim that we can identify  $\mathbb{P}_n(\mathcal{A})_f$  with the spectrum of some sheaf of algebras over  $Y$ . Namely, we have the surjection  $\mathcal{A}_1 \rightarrow \mathrm{gr}(\mathcal{A})_1$  and locally we can lift  $f$  to a section  $\tilde{f} \in \mathcal{A}_1$ . Consider the graded  $\mathcal{O}_Y$ -algebra

$$\mathrm{gr}_{(n)}(\mathcal{A})_f := \mathrm{gr}_{(n)}(\mathcal{A})_{\tilde{f}}.$$

It is easy to see that this algebra doesn't depend on a choice of the lifting element  $\tilde{f}$  (since two liftings differ by a nilpotent), hence this algebra is defined globally over  $Y$ . Now let  $\mathrm{gr}_{(n)}(\mathcal{A})_{(f)}$  be the degree zero component in  $\mathrm{gr}_{(n)}(\mathcal{A})_f$ . Then  $\mathrm{Spec}(\mathrm{gr}_{(n)}(\mathcal{A})_{(f)})$  is an open subscheme in  $\mathbb{P}_n(\mathcal{A})$  with the underlying open subset  $Y \times_S D_f$ , hence  $\mathrm{Spec}(\mathrm{gr}_{(n)}(\mathcal{A})_{(f)}) = \mathbb{P}_n(\mathcal{A})_f$ .

For a graded  $\mathrm{gr}_{(n)}(\mathcal{A})$ -module  $M$  we have the corresponding quasi-coherent sheaf  $\tilde{M}$  on  $\mathbb{P}_n(\mathcal{A})$ . The restriction of  $\tilde{M}$  to  $\mathbb{P}_n(\mathcal{A})_f$  is the sheaf associated with  $\mathrm{gr}_{(n)}(\mathcal{A})_{(f)}$ -module  $M_{(f)}$ , the degree zero part in the localization of  $M$  with respect to some local lifting of  $f$ .

Let as called a graded sheaf *graded special* if every of its graded components is a special sheaf.

**Lemma 4.1.** *For every element  $f \in C_1$  the algebra  $\mathrm{gr}_{(n)}(\mathcal{A})_f$  is a graded special D-algebra on  $Y$ . There is a canonical isomorphism of graded algebras*

$$(4.1.1) \quad \Phi(\mathrm{gr}_{(n)}(\mathcal{A})_f) \simeq \mathrm{gr}_{(n)}(\Phi\mathcal{A})_f.$$

*Proof.* First of all the Lemma is obvious for  $n = 0$ : in this case  $\mathrm{gr}(\mathcal{A})_f \simeq \pi_S^{Y*}(C_f)$  and

$$\Phi(\mathrm{gr}(\mathcal{A})_f) \simeq \mathrm{gr}(\Phi\mathcal{A})_f \simeq \pi_S^{X*}(C_f).$$

Now for  $n > 0$  consider the filtration on  $\mathrm{gr}_{(n)}(\mathcal{A})$  by two-sided principal ideals  $I^k = (t^k)$  where  $t \in \mathrm{gr}_{(n)}(\mathcal{A})_1$  is the central element corresponding to  $1 \in \mathcal{A}_1$ . Then  $I^0 = \mathcal{A}$ ,  $I^n = 0$ , and  $I^k/I^{k+1} \simeq \mathrm{gr}(\mathcal{A})(-k)$  as  $\mathrm{gr}_{(n)}(\mathcal{A})$ -module for  $0 \leq k < n$ . Localizing this filtration we get a filtration by two sided-ideals  $I_f^k$  in  $\mathrm{gr}_{(n)}(\mathcal{A})_f$  with associated graded quotients  $\mathrm{gr}(\mathcal{A})_{f(-k)}$ . Thus,  $\mathrm{gr}_{(n)}(\mathcal{A})_f$  is graded special.

To construct an isomorphism (4.1.1) we notice that since  $\Phi(\mathrm{gr}_{(n)}(\mathcal{A})_f)$  is a nilpotent extension of  $\Phi(\mathrm{gr}(\mathcal{A})_f) \simeq \mathrm{gr}(\Phi\mathcal{A})_f$ , any local lifting of  $f$  is invertible in  $\Phi(\mathrm{gr}_{(n)}(\mathcal{A})_f)$ . Therefore, by universal property we get a homomorphism

$$\mathrm{gr}_{(n)}(\Phi\mathcal{A})_f \rightarrow \Phi(\mathrm{gr}_{(n)}(\mathcal{A})_f).$$

Using the above filtration on  $\mathrm{gr}_{(n)}$  one immediately checks that this is an isomorphism.  $\square$

**Theorem 4.1.** *Assume that  $H^0(X, \mathcal{O}_X) = H^0(Y, \mathcal{O}_Y) = \mathbb{C}$ . Then for every  $n \geq 0$  there is a canonical equivalence of categories*

$$\Phi_{(n)} : \mathcal{D}^-(\mathbb{P}_n(\mathcal{A})) \rightarrow \mathcal{D}^-(\mathbb{P}_n(\Phi\mathcal{A}))$$

*commuting with functors  $i_{n*}$  and  $i_n^*$ . Moreover, assume that  $M$  is a left  $\mathcal{A}$ -module with an increasing module filtration such that for some integer  $d$  and for  $i$  sufficiently large  $\Phi(M_i/M_{i-1})$  is concentrated in degree  $d$  (as an object of  $\mathcal{D}(X)$ ). Then*

$$\mathrm{ml}_n(\Phi(M)) \simeq \Phi_{(n)}(\mathrm{ml}_n(M)).$$

*Proof.* The proof of this theorem is similar to the proof of Theorem 2.1. First we note that the definition of the operation  $\circ$  from 1.2 works for non-commutative schemes as well (the only difference is that now whenever we need to take the opposite  $D$ -algebra we have to pass to the opposite scheme as well). Now we just want to construct some quasi-coherent sheaves (perhaps shifted) on  $\mathbb{P}_n(\mathcal{A}) \times \mathbb{P}_n(\Phi\mathcal{A}^{op})$  and on  $\mathbb{P}_n(\Phi\mathcal{A}) \times \mathbb{P}_n(\mathcal{A}^{op})$  such that both their  $\circ$ -composition are equal to the structure sheaves of diagonals (note that although there is no embedding of a noncommutative scheme  $Z$  into  $Z \times Z^{op}$  we still can define an analogue of the structure sheaf of the diagonal  $\delta_Z$  which is a quasi-coherent sheaf on  $Z \times Z^{op}$ ). The definition of these sheaves is the following. First we observe that  $\mathbb{P}_n(\Phi\mathcal{A}) \times \mathbb{P}_n(\mathcal{A}^{op}) = \mathrm{Proj}(\mathcal{A}_{XY})$  where  $\mathcal{A}_{XY}$  is the following graded algebra on  $X \times Y$ :

$$\mathcal{A}_{XY} = \oplus_i \mathrm{gr}_{(n)}(\Phi\mathcal{A})_i \boxtimes \mathrm{gr}_{(n)}(\mathcal{A}^{op})_i.$$

Next we remark that the sheaf

$$\mathcal{B} = P \circ_{\mathcal{O}_Y} b(\mathcal{A}) = b(\Phi\mathcal{A}) \circ_{\mathcal{O}_X} P$$

introduced in the proof of Theorem 2.1 has a natural filtration

$$\mathcal{B}_i = P \circ_{\mathcal{O}_Y} b(\mathcal{A})_i = b(\Phi\mathcal{A})_i \circ_{\mathcal{O}_X} P.$$

It follows that the sheaf  $\oplus_i \mathcal{B}_{2i}/\mathcal{B}_{2i-n}$  has a natural structure of graded  $\mathcal{A}_{XY}$ -module, so we can set

$$\mathcal{B}_{ml} = (\oplus_i \mathcal{B}_{2i}/\mathcal{B}_{2i-n})^\sim$$

which is a quasi-coherent sheaf on  $\mathbb{P}_n(\Phi\mathcal{A}) \times \mathbb{P}_n(\mathcal{A}^{op})$ . Similarly, one can define the quasi-coherent sheaf  $\mathcal{B}'_{ml}$  on  $\mathbb{P}_n(\mathcal{A}) \times \mathbb{P}_n(\Phi\mathcal{A}^{op})$  starting with the sheaf  $\mathcal{B}' = Q \circ_{\mathcal{O}_X} b(\Phi\mathcal{A})$ .

It remains to compute  $\mathcal{B}_{ml} \circ_{\mathcal{O}_{\mathbb{P}_n(\mathcal{A})}} \mathcal{B}'_{ml}$  and  $\mathcal{B}'_{ml} \circ_{\mathcal{O}_{\mathbb{P}_n(\Phi\mathcal{A})}} \mathcal{B}_{ml}$ . The idea is the following: we cover  $\mathrm{Proj}(C)$  by affine subsets  $D_f$ , where  $f$  runs through  $C_1$ . For every  $f \in C_1$  we'll construct a canonical isomorphism between restrictions of  $\mathcal{B}_{ml} \circ_{\mathcal{O}_{\mathbb{P}_n(\mathcal{A})}} \mathcal{B}'_{ml}$  and the structure sheaf of diagonal in  $\mathbb{P}_n(\Phi\mathcal{A}) \times \mathbb{P}_n(\Phi\mathcal{A}^{op})$  to the open subscheme  $\mathbb{P}_n(\Phi\mathcal{A})_f \times \mathbb{P}_n(\Phi\mathcal{A}^{op})_f$ . These isomorphisms will be compatible on intersections, so they will glue into a global isomorphism.

The following notation will be useful: for a sheaf  $\mathcal{F}$  on one of our schemes and an element  $f \in C_1$  we denote by  $\mathcal{F}_f$  the restriction of  $\mathcal{F}$  to the open subscheme defined by  $f$ .

Under identification of the underlying topological space of  $\mathbb{P}_n(\Phi\mathcal{A}) \times \mathbb{P}_n(\mathcal{A}^{op})$  with  $X \times \text{Proj}(C) \times Y \times \text{Proj}(C)$  the support of  $\mathcal{B}_{ml}$  is the diagonal  $X \times Y \times \text{Proj}(C)$ . Using this fact it is fairly easy to see that

$$(\mathcal{B}_{ml} \circ_{\mathcal{O}_{\mathbb{P}_n(\mathcal{A})}} \mathcal{B}'_{ml})_f = \mathcal{B}_{ml,f} \circ_{\mathcal{O}_{\mathbb{P}_n(\mathcal{A})_f}} \mathcal{B}'_{ml,f}$$

It remains to compute the  $\circ$ -composition in the RHS. This is easier than the original problem because the sheaf  $\mathcal{B}_{ml,f}$  (resp.  $\mathcal{B}'_{ml,f}$ ) live on affine schemes over  $X \times Y$  (resp.  $Y \times X$ ). Namely,

$$\mathbb{P}_n(\Phi\mathcal{A})_f \times \mathbb{P}_n(\mathcal{A}^{op})_f = \text{Spec}(\text{gr}_{(n)}(\Phi\mathcal{A})_{(f)} \boxtimes \text{gr}_{(n)}(\mathcal{A}^{op})_{(f)}).$$

According to Lemma 4.1 we have dual  $D$ -algebras  $\text{gr}_{(n)}(\mathcal{A})_{(f)}$  on  $Y$  and  $\text{gr}_{(n)}(\Phi\mathcal{A})_{(f)}$  on  $X$ . Hence, we can apply Theorem 2.1 to these  $D$ -algebras. Let us denote by  $\mathcal{B}_{(f)}$  the  $\text{gr}_{(n)}(\Phi\mathcal{A})_{(f)} \boxtimes \text{gr}_{(n)}(\mathcal{A}^{op})_{(f)}$ -module constructed in the proof of the cited theorem (where it is called  $\mathcal{B}$ ). We claim that there is a canonical isomorphism of the  $\mathcal{B}_{ml,f}$  with the sheaf on  $\text{Spec}(\text{gr}_{(n)}(\Phi\mathcal{A})_{(f)} \boxtimes \text{gr}_{(n)}(\mathcal{A}^{op})_{(f)})$  obtained by localization of  $\mathcal{B}_{(f)}$ . This claim (together with an easy check of the compatibility of isomorphisms on intersections) would allow to finish the proof by referring to Theorem 2.1. It remains to construct an isomorphism between the two  $\text{gr}_{(n)}(\Phi\mathcal{A})_{(f)} \boxtimes \text{gr}_{(n)}(\mathcal{A}^{op})_{(f)}$ -modules:  $\mathcal{B}_{(f)}$  and  $(\oplus_i \mathcal{B}_{2i}/\mathcal{B}_{2i-n})_{(f \otimes f)}$  (the localization of the latter module is clearly  $\mathcal{B}_{ml,f}$ ). Recall that by definition  $\mathcal{B}_{(f)} = P \circ_{\mathcal{O}_Y} \text{gr}_{(n)} \mathcal{A}_{(f)}$ . Also, it is clear that  $(\oplus_i \mathcal{B}_{2i}/\mathcal{B}_{2i-n})_{(f \otimes f)}$  is isomorphic to the degree zero part in the localization of

$$\text{gr}_{(n)}(\mathcal{B}) = \oplus_i \mathcal{B}_i/\mathcal{B}_{i-n} = P \circ_{\mathcal{O}_Y} \text{gr}_{(n)}(\mathcal{A})$$

by  $\tilde{f} \otimes 1$  and  $1 \otimes \tilde{f}'$ , where  $\tilde{f}$  is a local lifting of  $f$  to  $\Phi\mathcal{A}_1$ ,  $\tilde{f}'$  is a local lifting of  $f$  to  $\mathcal{A}_1$ . Thus, it suffices to construct a graded isomorphism between  $P \circ_{\mathcal{O}_Y} \text{gr}_{(n)}(\mathcal{A})_f$  and  $\text{gr}_{(n)}(\mathcal{B})_{\tilde{f} \otimes 1, 1 \otimes \tilde{f}'}$ . According to Lemma 4.1 we have

$$P \circ_{\mathcal{O}_Y} \text{gr}_{(n)}(\mathcal{A})_f \simeq \text{gr}_{(n)}(\Phi\mathcal{A})_f \circ_{\mathcal{O}_X} P$$

so the assertion follows.  $\square$

Note that we have canonical invertible  $\mathcal{O}_{\mathbb{P}_n}$ -bimodules on  $\mathbb{P}_n(\mathcal{A})$ :

$$\mathcal{O}_{\mathbb{P}_n}(m) = \text{ml}_n(\text{gr}_{(n)}(\mathcal{A})(m))$$

where  $M \mapsto M(m)$  denotes the shift of grading. In particular, we have the automorphism

$$M \mapsto M(1) = \mathcal{O}(1) \otimes_{\mathcal{O}} M$$

of the category  $\mathcal{D}^-(\mathbb{P}_n(\mathcal{A}))$ . It is easy to see that the above equivalence respects these automorphisms.

**4.2.** One can generalize Theorem 2.1 to the case of NC-schemes of finite degree. Namely, there is a natural notion of support of a quasi-coherent sheaf on such a scheme (just the support of the corresponding sheaf on the reduced commutative scheme), hence, the definition of  $D$ -algebra makes sense. Now the proof of Theorem 2.1 works almost literally in this case. Moreover, it seems plausible for the NC-schemes  $\mathbb{P}_n(\mathcal{A})$  one can consider slightly more general  $D$ -algebras than special ones. Namely, instead of requiring the existence of filtration with graded factors isomorphic to  $\mathcal{O}$  it suffices to require the existence of filtration with factors  $\mathcal{O}^{ab}$ , plus one should require  $D$ -algebra to be flat as left and right  $\mathcal{O}$ -module.

**4.3. Étale local version of the equivalence.** Let  $U \rightarrow Z$  be an étale morphism of  $S$ -schemes. Then we have the corresponding étale morphism  $Y \times U \rightarrow \mathbb{P}_1(\mathcal{A})$ . By topological invariance of étale category for every  $n \geq 1$  this morphism extends to an étale morphism of NC-schemes

$$j : \mathbb{P}_n(\mathcal{A})_U \rightarrow \mathbb{P}_n(\mathcal{A}).$$

Similarly we have an NC-scheme  $\mathbb{P}_n(\Phi\mathcal{A})_U$ , and an étale morphism  $j : \mathbb{P}_n(\Phi\mathcal{A})_U \rightarrow \mathbb{P}_n(\Phi\mathcal{A})$ .

**Theorem 4.2.** *In the above situation the categories  $\mathcal{D}^-(\mathbb{P}_n(\mathcal{A})_U)$  and  $\mathcal{D}^-(\mathbb{P}_n(\Phi\mathcal{A})_U)$  are canonically equivalent.*

*Proof.* Recall that in the proof of Theorem 4.1 we have constructed a quasi-coherent sheaf (up to shift)  $\mathcal{B}_{ml}$  on  $\mathbb{P}_n(\Phi\mathcal{A}) \times \mathbb{P}_n(\mathcal{A}^{op})$  supported on the diagonal  $\Delta_Z : X \times Y \times Z \hookrightarrow X \times Z \times Y \times Z$ . Moreover, the restriction of  $\mathcal{B}_{ml}$  to  $\mathbb{P}_1(\Phi\mathcal{A}) \times \mathbb{P}_n(\mathcal{A}^{op})$  is actually obtained from the sheaf  $P$  on  $X \times Y$  via first pulling back to  $X \times Y \times Z$  and then pushing forward by  $\Delta_Z$ . We have the following diagram of étale morphisms of NC-schemes:

$$(4.3.1) \quad \begin{array}{ccc} \mathbb{P}_n(\Phi\mathcal{A})_U \times \mathbb{P}_n(\mathcal{A}^{op})_U & \xrightarrow{\text{id} \times j} & \mathbb{P}_n(\Phi\mathcal{A})_U \times \mathbb{P}_n(\mathcal{A}^{op}) \\ \downarrow j \times \text{id} & & \downarrow j \times \text{id} \\ \mathbb{P}_n(\Phi\mathcal{A}) \times \mathbb{P}_n(\mathcal{A}^{op})_U & \xrightarrow{\text{id} \times j} & \mathbb{P}_n(\Phi\mathcal{A}) \times \mathbb{P}_n(\mathcal{A}^{op}) \end{array}$$

Now we claim that there exists a quasi-coherent sheaf  $\mathcal{B}_{ml,U}$  on  $\mathbb{P}_n(\Phi\mathcal{A})_U \times \mathbb{P}_n(\mathcal{A}^{op})_U$  supported on the diagonal  $\Delta_U : X \times Y \times U \hookrightarrow X \times U \times Y \times U$  such that

$$(4.3.2) \quad (\text{id} \times j)_* \mathcal{B}_{ml,U} \simeq (j \times \text{id})^* \mathcal{B}_{ml}$$

$$(4.3.3) \quad (j \times \text{id})_* \mathcal{B}_{ml,U} \simeq (\text{id} \times j)^* \mathcal{B}_{ml}.$$

and such that the restriction of  $\mathcal{B}_{ml,U}$  to  $\mathbb{P}_1(\Phi\mathcal{A})_U \times \mathbb{P}_1(\mathcal{A})$  is isomorphic to  $\Delta_{U,*}(p_{XY}^* P)$ . Indeed, consider the quasi-coherent sheaf  $(j \times j)^* \mathcal{B}_{ml}$  on  $\mathbb{P}_n(\Phi\mathcal{A})_U \times \mathbb{P}_n(\mathcal{A}^{op})_U$ . It is supported on  $(j \times j)^{-1}(X \times Y \times Z)$  where  $X \times Y \times Z$  is the relative diagonal in  $X \times Y \times Z \times Z$ . Now since  $j$  is étale, the relative diagonal  $X \times Y \times U$  is a connected component in  $(j \times j)^{-1}(X \times Y \times Z)$ . Now we just set  $\mathcal{B}_{ml,U}$  to be the direct summand of  $(j \times j)^* \mathcal{B}_{ml}$  concentrated on this component, i.e.

$$\mathcal{B}_{ml,U} = (j \times j)^* \mathcal{B}_{ml}|_{X \times Y \times U}.$$

The above properties of  $\mathcal{B}_{ml,U}$  are clear from this definition.

Similarly, we construct the sheaf  $\mathcal{B}'_{ml,U}$  on  $\mathbb{P}_n(\mathcal{A}) \times \mathbb{P}_n(\Phi\mathcal{A}^{op})$ . It remains to compute the relevant  $\circ$ -products. This is easily done using isomorphisms (4.3.2), (4.3.3). Namely, one should start by computing  $(j \times \text{id})_*(\mathcal{B}_{ml,U} \circ_{\mathbb{P}_n(\mathcal{A})_U} \mathcal{B}'_{ml,U})$  on  $\mathbb{P}_n(\Phi\mathcal{A}) \times \mathbb{P}_n(\Phi\mathcal{A}^{op})_U$ . We have

$$\begin{aligned} (j \times \text{id})_*(\mathcal{B}_{ml,U} \circ_{\mathbb{P}_n(\mathcal{A})_U} \mathcal{B}'_{ml,U}) &\simeq ((j \times \text{id})_* \mathcal{B}_{ml,U}) \circ_{\mathbb{P}_n(\mathcal{A})_U} \mathcal{B}'_{ml,U} \simeq ((\text{id} \times j)^* \mathcal{B}_{ml}) \circ_{\mathbb{P}_n(\mathcal{A})_U} \mathcal{B}'_{ml,U} \simeq \\ &\mathcal{B}_{ml} \circ_{\mathbb{P}_n(\mathcal{A})} ((j \times \text{id})_* \mathcal{B}'_{ml,U}) \simeq \mathcal{B}_{ml} \circ_{\mathbb{P}_n(\mathcal{A})} ((\text{id} \times j)^* \mathcal{B}'_{ml}) \simeq (\text{id} \times j)^*(\mathcal{B}_{ml} \circ_{\mathbb{P}_n(\mathcal{A})} \mathcal{B}'_{ml}) \simeq \\ &(\text{id} \times j)^*(\delta_{\mathbb{P}_n(\Phi\mathcal{A})}) \simeq (j \times \text{id})_*(\delta_{\mathbb{P}_n(\Phi\mathcal{A})_U}) \end{aligned}$$

Now the situation looks locally as follows: we have an étale extension of NC-algebras  $A \rightarrow A_1$ , an  $A_1 \otimes A_1^{op}$ -module  $M$ , and an isomorphism of  $A \otimes A_1^{op}$ -modules  $M \simeq A_1$ . Furthermore, we have a 2-sided ideal  $I \subset A$  such that  $A/I$  is commutative and  $IM = MI$ , and we know that the induced isomorphism  $M/IM \simeq A_1/IA_1$  is an isomorphism of  $A_1/IA_1 \otimes A_1/IA_1$ -modules (notice that  $A_1/IA_1$  is commutative). We claim that in such a situation the above isomorphism commutes with the left action of  $A_1$ . Indeed, the left action of  $A_1$  on  $M$  induces a homomorphism  $\phi : A_1 \rightarrow A_1$  such that  $\phi|_A = \text{id}$  and  $\phi \bmod IA_1$  is the identity on  $A_1/IA_1$ . Now the formal étaleness implies that  $\phi = \text{id}$ .

Thus, we conclude that  $\mathcal{B}_{ml,U} \circ_{\mathbb{P}_n(\mathcal{A})_U} \mathcal{B}'_{ml,U} \simeq \delta_{\mathbb{P}_n(\Phi\mathcal{A})_U}$  as required.  $\square$

**4.4.** The sheaf of rings  $\mathcal{O}_{\mathbb{P}_n}$  on  $\mathbb{P}_n$  can be naturally enlarged as follows. The central element  $t \in \text{gr}_{(n)}(\mathcal{A})_1$  induces a sequence of embeddings of  $\mathcal{O}_{\mathbb{P}_n}$ -bimodules

$$\mathcal{O}_{\mathbb{P}_n} \rightarrow \mathcal{O}_{\mathbb{P}_n}(1) \rightarrow \mathcal{O}_{\mathbb{P}_n}(2) \rightarrow \dots$$

Now using the natural morphisms

$$\mathcal{O}_{\mathbb{P}_n}(m) \otimes_{\mathcal{O}_{\mathbb{P}_n}} \mathcal{O}_{\mathbb{P}_n}(l) \rightarrow \mathcal{O}_{\mathbb{P}_n}(m+l)$$

we can define the ring structure on the direct limit

$$\tilde{\mathcal{O}}_{\mathbb{P}_n(\mathcal{A})} = \text{inj. lim}(\mathcal{O} \rightarrow \mathcal{O}(1) \rightarrow \mathcal{O}(2) \rightarrow \dots)$$

For example, if  $Y$  is smooth and  $\mathcal{A} = D_Y$  is the sheaf of differential operators on  $Y$  then  $\tilde{\mathcal{O}}_{\mathbb{P}_\infty}$  is the sheaf of (formal) pseudo-differential operators (the underlying topological space of  $\mathbb{P}_\infty$  is the projectivized cotangent bundle of  $Y$ ). The subsheaf  $\mathcal{O}_{\mathbb{P}_\infty}$  consists of (formal) pseudo-differential operators of negative order.

Now let  $\mathbb{P}_1(\mathcal{A}) = Y \times Z$  and  $U \rightarrow Z$  be an étale morphism. Then one can define invertible  $\mathcal{O}_{\mathbb{P}_n(\mathcal{A})_U}$ -bimodules  $\mathcal{O}_{\mathbb{P}_n(\mathcal{A})_U}(m)$  as follows. We can regard  $\mathcal{O}_{\mathbb{P}_n(\mathcal{A})}$  as a sheaf on  $\mathbb{P}_n(\mathcal{A}) \times \mathbb{P}_n(\mathcal{A}^{op})$  supported on the diagonal. Let  $V$  be a thickening of the diagonal in  $\mathbb{P}_n(\mathcal{A}) \times \mathbb{P}_n(\mathcal{A}^{op})$  on which  $\mathcal{O}_{\mathbb{P}_n(\mathcal{A})}(m)$  lives. Then there is a canonical étale morphism  $V_U \rightarrow V$  and an embedding  $V_U \rightarrow \mathbb{P}_n(\mathcal{A})_U \times \mathbb{P}_n(\mathcal{A}^{op})_U$ . Now by definition  $\mathcal{O}_{\mathbb{P}_n(\mathcal{A})_U}(m)$  is obtained from  $\mathcal{O}_{\mathbb{P}_n(\mathcal{A})}(m)$  by first pulling back to  $V_U$  and then pushing forward to  $\mathbb{P}_n(\mathcal{A})_U \times \mathbb{P}_n(\mathcal{A}^{op})_U$ . It is easy to see that we still have morphisms of bimodules  $\mathcal{O}(m) \rightarrow \mathcal{O}(m+1)$  and  $\mathcal{O}(n) \otimes \mathcal{O}(m) \rightarrow \mathcal{O}(n+m)$  so we can define the algebra  $\tilde{\mathcal{O}}_{\mathbb{P}_n(\mathcal{A})_U}$ .

**Theorem 4.3.** *In the preceding two theorems one can replace the categories of  $\mathcal{O}$ -modules by the categories of  $\tilde{\mathcal{O}}$ -modules.*

The proof is an application of the analogue of Theorem 2.1 for  $D$ -modules on NC-schemes.

## 5. NONCOMMUTATIVE DEFORMATION OF THE POINCARÉ LINE BUNDLE

Consider the following data:

$W$  is a smooth projective variety over  $\mathbb{C}$  of dimension  $r$ ,

$D \subset W$  is a reduced irreducible effective divisor,

$V \subset H^0(D, \mathcal{O}_D(D))$  is an  $r$ -dimensional subspace, such that the corresponding rational morphism  $\phi : D \rightarrow \mathbb{P}(V^*)$  is generically finite,

$U \subset D$  is an open subset such that  $\phi|_U$  is étale.

From the exact sequence

$$0 \rightarrow \mathcal{O}_W \rightarrow \mathcal{O}_W(D) \rightarrow \mathcal{O}_D(D) \rightarrow 0$$

we get a boundary homomorphism

$$V \rightarrow H^0(D, \mathcal{O}_D(D)) \rightarrow H^1(W, \mathcal{O}_W).$$

Now let  $X$  be the Albanese variety of  $W$ ,  $a : W \rightarrow X$  be the Abel-Jacobi map (associated with some point of  $W$ ). Then we have the canonical isomorphism

$$H^1(X, \mathcal{O}_X) \xrightarrow{\sim} H^1(W, \mathcal{O}_W),$$

in particular, we get a homomorphism  $V \rightarrow H^1(W, \mathcal{O}_W)$ . Let

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{E} \rightarrow H^1(X, \mathcal{O}_X) \otimes_{\mathbb{C}} \mathcal{O}_X \rightarrow 0$$

be the universal extension. Taking the pull-back of this extension under the map  $V \rightarrow H^1(W, \mathcal{O}_W)$  we obtain an extension

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{E}_V \rightarrow V \otimes_{\mathbb{C}} \mathcal{O}_X \rightarrow 0.$$

Now we define a commutative sheaf of algebras on  $X$  as follows

$$\mathcal{A}_V = \text{Sym}(\mathcal{E}_V)/(1_{\mathcal{E}} - 1)$$

where  $1_{\mathcal{E}}$  is the image of  $1 \in \mathcal{O}_X \rightarrow \mathcal{E}_V$ . Note that  $\mathcal{A}_V$  is equipped with the filtration satisfying the conditions of the previous section. Also by construction we have a canonical morphism of sheaves

$$\mathcal{E}_V \rightarrow a_* \mathcal{O}_W(D)$$

which induce the homomorphism of  $\mathcal{O}_X$ -algebras

$$\mathcal{A}_V \rightarrow a_* \mathcal{O}_W(*D)$$

compatible with natural filtrations, where  $\mathcal{O}_W(*D) = \text{inj. lim } \mathcal{O}_W(nD)$ .

If the map  $V \rightarrow H^1(W, \mathcal{O}_W)$  is an embedding then the dual  $D$ -algebra  $\Phi \mathcal{A}_V$  is the algebra of differential operators “in directions  $V$ ”, where we consider  $V$  as a subspace in  $H^1(X, \mathcal{O}_X) \simeq H^0(\hat{X}, T_{\hat{X}})$ . By Theorem 2.1 we get a functor from the derived category of  $\mathcal{A}_V$ -modules to the derived category of  $\Phi \mathcal{A}_V$ -modules. We can restrict this functor to the category of  $\mathcal{O}_W(*D)$ -modules. For example, if  $D$  is ample the Fourier transform of  $\mathcal{O}_W(*D)$  is a coherent  $\Phi \mathcal{A}_V$ -module. In the case when  $W$  is a curve,  $D$  is a point, and  $a(D) = 0 \in X$  one can show that the latter  $\Phi \mathcal{A}_V$ -module is free of rank 1 at general point. However, in general  $\mathcal{F}(\mathcal{O}_W(*D))$  is

not free as  $\Phi\mathcal{A}_V$ -module even at general point unless  $a(D) = 0$ . To get a module which is free of rank 1 at general point we have to pass to microlocalization and use étale localization "in vector fields direction" as described below.

Let  $D_{(n)} \subset W$  be the closed subscheme corresponding to the divisor  $nD$ . Then  $\text{Proj}(\text{gr}_{(n)}(\mathcal{O}_W(*D))) \simeq D_{(n)}$ , hence by functoriality we have a morphism

$$a_n : D_{(n)} \rightarrow \mathbb{P}_n(\mathcal{A}_V)$$

and an isomorphism

$$a_{n,*}\mathcal{O}_{D_{(n)}} \simeq \text{ml}_n(\mathcal{O}_W(*D))$$

where  $\mathcal{O}_W(*D)$  is considered as a  $\mathcal{A}_V$ -module.

Let us start with the case  $n = 1$ . Note that

$$a_1 : D \rightarrow \mathbb{P}_1(\mathcal{A}_V) \simeq X \times \mathbb{P}(V^*)$$

is the natural map induces by  $a$  and by  $\phi$ . Hence, applying Fourier-Mukai transform to  $a_{1,*}\mathcal{O}_D$  over a general point of  $\mathbb{P}(V^*)$  one gets a free module of rank equal to the degree of  $\phi$ . To get a free module of rank 1 at general point we use the étale base change  $U \rightarrow \mathbb{P}(V^*)$ . Namely, we replace  $\mathbb{P}_1(\mathcal{A}_V) = X \times \mathbb{P}(V^*)$  by  $\mathbb{P}_1(\mathcal{A}_V)_U = X \times U$  and  $a_1$  by the morphism

$$a_1^U : U \rightarrow \mathbb{P}_1(\mathcal{A}_V)_U = X \times U : u \mapsto (a(u), u).$$

Then the following lemma is clear.

**Lemma 5.1.** *The relative Fourier transform of  $a_{1,*}^U\mathcal{O}_U$  is the line bundle  $(id \times a|_U)^*(\mathcal{P})$  on  $\hat{X} \times U$ , where  $\mathcal{P}$  is the Poincaré line bundle.*

Now let  $\mathbb{P}_n(\mathcal{A}_V)_U$  be the étale scheme over  $\mathbb{P}_n(\mathcal{A}_V)$  which is a thickening of  $X \times U$ . Let also  $U_{(n)}$  be the open subset of  $D_{(n)}$  which is a nilpotent thickening of  $U$ . Then we have a commutative diagram

$$(5.0.1) \quad \begin{array}{ccc} U & \longrightarrow & \mathbb{P}_n(\mathcal{A}_V)_U \\ \downarrow & & \downarrow \\ U_{(n)} & \longrightarrow & \mathbb{P}_n(\mathcal{A}_V) \end{array}$$

where the top horizontal arrow is the composition of  $a_1^U$  and the closed embedding  $\mathbb{P}_1(\mathcal{A}_V)_U \rightarrow \mathbb{P}_n(\mathcal{A}_V)_U$ , the bottom horizontal arrow is the restriction of  $a_n$ . Since the right vertical morphism is étale this diagram gives rise to a morphism

$$a_n^U : U_{(n)} \rightarrow \mathbb{P}_n(\mathcal{A}_V)_U$$

filling the diagonal in the above commutative square. It is easy to check that  $a_n^U|_{U_{(n-1)}} = a_{n-1}^U$ . Now we define the sequence of coherent sheaves on  $\mathbb{P}_n(\Phi\mathcal{A}_V)_U$  by setting

$$\mathcal{L}_n = \Phi_{(n)}(a_{n,*}^U\mathcal{O}_{U_{(n)}}).$$

These sheaves satisfy  $\mathcal{L}_{n+1}|_{\mathbb{P}_n} \simeq \mathcal{L}_n$ , hence we can consider the projective limit  $\mathcal{L}_\infty$  of  $\mathcal{L}_n$  which is a coherent sheaf on the formal NC-scheme  $\mathbb{P}_\infty(\Phi\mathcal{A}_V)_U$ .

**Proposition 5.1.** *The  $\mathcal{O}_{\mathbb{P}_\infty(\Phi_{\mathcal{A}_V})_U}$ -module  $\mathcal{L}_\infty$  is locally free of rank 1.*

*Proof.* One has an exact sequence

$$0 \rightarrow a_{n-1,*}^U \mathcal{O}_{U_{(n-1)}}(-1) \rightarrow a_{n,*}^U \mathcal{O}_{U_{(n)}} \rightarrow a_{1,*}^U \mathcal{O}_U \rightarrow 0.$$

Applying the functor  $\Phi_{(n)}$  and passing to the limit we obtain an exact sequence

$$0 \rightarrow \mathcal{L}_\infty(-1) \xrightarrow{t} \mathcal{L}_\infty \rightarrow \mathcal{L}_1 \rightarrow 0$$

where  $t$  is the canonical central element in  $\mathcal{O}(1)$ . It remains to use the following simple algebraic fact. Let  $A$  be a noetherian ring,  $t \in A$  be a non zero divisor such that  $At = tA$  and  $A = \text{proj. lim } A/t^n A$ . Let  $M$  be a finitely generated left  $A$ -module such that  $t$  is not a divisor of zero in  $M$  and  $M/tM$  is a free  $A/tA$ -module of rank 1. Then  $M$  is a free  $A$ -module of rank 1.  $\square$

Notice that in the case when  $W = C$  is a curve  $D = P$  is a point,  $U = D$  the module  $\mathcal{L}_\infty$  was used in [R1] to construct Krichever's solution to the KP hierarchy. In this case  $\mathcal{L}_\infty$  is a locally free module of rank-1 over the microlocalization of the subalgebra  $\mathcal{O}[\xi]$  in the ring of the differential operators on the Jacobian  $J(C)$  generated by the vector field  $\xi$  which comes from the boundary homomorphism  $H^0(\mathcal{O}_P(P)) \rightarrow H^1(\mathcal{O}_C)$ . The key point is that  $\mathcal{L}_\infty$  has also an action of completion of  $\mathcal{O}_C(*P)$  at  $P$  (which is isomorphic to the ring of Laurent series) commuting with the action of pseudo-differential operators in  $\xi$ .

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